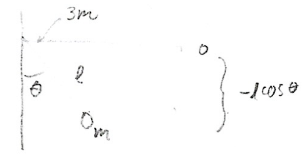
CM-2 January 2012

Pendulum mass m

Rigid rod length l $\rightarrow$ constant density $\rho = \frac{M}{l}$

mass $3m$



Moment inertia about C.M $= \frac{Ml^2}{12}$

Parallel axis theorem $I = I_{cm} + ML^2 = \frac{Ml^2}{12} + \left(\frac{l}{2}\right)^2 M = Ml^2 \underbrace{\left(\frac{1}{12} + \frac{1}{4}\right)}_{\frac{4}{12}} = \frac{1}{3} ML^2$ about end

a) Find Lagrange Equation for θ

$T = T_{Ball} + T_{rod}$

$T_{Ball} = \frac{1}{2} m v^2 = \frac{1}{2} m r^2 \dot\theta^2$

$T_{rod} = \frac{1}{2} I \omega^2 = \frac{1}{2} I \dot\theta^2$

$v = r\omega$

$T = \frac{1}{2} m r^2 \dot\theta^2 + \frac{1}{2} I \dot\theta = \frac{1}{2} m l^2 \dot\theta^2 + \frac{1}{6} M l^2 \dot\theta^2 = \frac{1}{2} m l^2 \dot\theta^2 + \frac{1}{2} m l^2 \dot\theta^2 = m l^2 \dot\theta^2$

$U = U_{Ball} + U_{rod} = mgh_1 + Mgh_2 = mg\, l\cos\theta + 3mg \frac{l}{2} \cos\theta$

$= \frac{-5}{2} mgl \cos\theta$ $\rightarrow$ negative sign from being below zero in my coordinate system.

$\mathcal{L} = T - U = ml^2 \dot\theta^2 + \frac{5}{2} mgl \cos\theta$

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = \frac{\partial \mathcal{L}}{\partial \theta}$ $\Rightarrow$ $\frac{d}{dt}(2ml^2\dot\theta) = 2ml^2\ddot\theta$

$\frac{\partial \mathcal{L}}{\partial \theta} = -\sin\theta \cdot \left(\frac{5}{2} mgl\right)$

$\Rightarrow \ddot\theta = \frac{-5}{4} \frac{g}{l} \sin\theta$

b. Write down equation when θ is small (1st order approximation)

$\sin\theta \approx \theta + O(\theta^2)$

$\ddot\theta = -\frac{5}{4} \frac{g}{l} \theta$

c. Assume pendulum is in viscous medium. $3\sqrt{\frac{g}{\ell}}\frac{d\theta}{dt} \rightarrow 3\sqrt{\frac{g}{\ell}}\theta$

$\ddot{\theta} + 3\sqrt{g/\ell}\,\dot{\theta} = -\frac{5}{4}\frac{g}{\ell}\theta$ $\qquad t=0,\ \theta=0 \ \&\ \dot{\theta} = 2\sqrt{g/\ell}$

Find $\theta(t)$

Ansatz: $Ae^{\alpha t} \rightarrow \dot{\theta} = A\alpha e^{\alpha t} \ \&\ \ddot{\theta} = A\alpha^2 e^{\alpha t}$

$$\Rightarrow A\alpha^2 e^{\alpha t} + \sqrt{\frac{9g}{\ell}}A\alpha e^{\alpha t} = \frac{5}{4}\frac{g}{\ell}Ae^{\alpha t}$$

$$\alpha^2 + \sqrt{\frac{9g}{\ell}}\alpha = \frac{5}{4}\frac{g}{\ell} \qquad \alpha^2 + \sqrt{\frac{9g}{\ell}}\alpha + \frac{5}{4}\frac{g}{\ell} = 0$$

$$\frac{-\sqrt{\frac{9g}{\ell}} \pm \sqrt{\frac{9g}{\ell} + 4\frac{5}{4}\frac{g}{\ell}}}{2} = -\sqrt{\frac{9g}{4\ell}} \pm \frac{1}{2}\sqrt{\frac{(9-5)g}{\ell}}$$

$$\alpha = -\sqrt{\frac{9g}{4\ell}} \pm \sqrt{\frac{4g}{4\ell}} = -\frac{3}{2}\sqrt{\frac{g}{\ell}} \pm \sqrt{\frac{g}{\ell}}$$

$$\alpha_+ = -\frac{1}{2}\sqrt{\frac{g}{\ell}}$$
$$\alpha_- = -\frac{5}{2}\sqrt{\frac{g}{\ell}}$$

$\theta(t) = A_1 e^{\alpha_+ t} + A_2 e^{\alpha_- t}$ $\rightarrow$ At $t=0$: $0 = A_1 + A_2$ $\qquad A_1 = -A_2$

$\dot{\theta}(t) = \alpha_+ A_1 e^{\alpha_+ t} + A_2\alpha_- e^{\alpha_- t}$

At $t=0$:

$$2\sqrt{\frac{g}{\ell}} = \frac{-1}{2}\sqrt{\frac{g}{\ell}}A_1 + \frac{-5}{2}\sqrt{\frac{g}{\ell}}A_2 \Rightarrow 2 = -\frac{1}{2}A_1 - \frac{5}{2}A_2 \rightarrow 4 = -A_1 - 5A_2$$

$4 = A_2 - 5A_2 = -4A_2$ $\qquad A_2 = -1 \ \&\ A_1 = 1$

$$\theta(t) = e^{-\frac{1}{2}\sqrt{g/\ell}\,t} - e^{-\frac{5}{2}\sqrt{g/\ell}\,t}$$

Is motion under, critically or over damped?

$\rightarrow$ this depends on value of square root $\Rightarrow \underbrace{\sqrt{b^2-4ac}}_{\lambda}$

$\lambda > 0$ over-damped

$\lambda = 0$ critically damped

$\lambda < 0$ under-damped

$\Rightarrow \lambda = \frac{g}{\ell} > 0$

$\rightarrow$ the oscallator is over-damped