CM-2 January 2013

$\vec{S}(t)^2 = s^2$

$\dot{\vec{S}}(t) = \vec{S} \times \vec{B}$ $\rightarrow$ motion of spin in constant $\vec{B}$ field — Bloch Equation

a. Use spherical coordinates for $\vec{S}$ to work out $\theta(t)$ and $\phi(t)$

$\vec{S}(t) = S_r(t)\hat{r} + S_\theta(t)\hat{\theta} + S_\phi(t)\hat{\phi}$

$\vec{B}(t) = a\hat{r} + b\hat{\theta} + c\hat{\phi}$

Cross-product in sperical coordinates:

$$\vec{S}\times\vec{B} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ S_r(t) & S_\theta(t) & S_\phi(t) \\ a & b & c \end{vmatrix} = \left(S_\theta(t)B_\phi - S_\phi(t)B_\theta\right)\hat{r} - \left(S_r(t)B_\phi - S_\phi(t)B_r\right)\hat{\theta} + \left(S_r(t)B_\theta - S_\theta(t)B_r\right)$$

$\dot{\vec{S}}(t)\cdot\hat{r} = S_\theta(t)B_\phi - S_\phi(t)B_\theta$

$\dot{\vec{S}}(t)\cdot\hat{\theta} = S_\phi(t)B_r - S_r(t)B_\phi$ $\rightarrow$ If $\vec{B}$ is in $+\hat{z}$

$\dot{\vec{S}}(t)\cdot\hat{\phi} = S_r(t)B_\theta - S_\theta(t)B_r$ $\quad B_\phi = B_\theta = 0$

$\Rightarrow \dot{\vec{S}}_r(t) = 0$

$\dot{\vec{S}}_\theta(t) = S_\phi(t)B$ $\rightarrow$ $\dot{\theta}(t) = \phi B$ $\quad \dot{x} = yB$

$\dot{\vec{S}}_\phi(t) = -S_\theta(t)B$ $\quad \dot{\phi}(t) = -\theta B$ $\quad \dot{y} = -xB$

b. Find a Lagrangian for these equations

$\mathcal{L} = T - U$ $\qquad \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \dfrac{\partial \mathcal{L}}{\partial q_i}$

$\dot{\theta} \rightarrow \dfrac{\partial \mathcal{L}}{\partial \dot{q}_1} = \theta$ $\quad \& \quad \dfrac{\partial \mathcal{L}}{\partial q_1} = \phi B$

$\mathcal{L} = \theta\dot{\phi} + \phi^2 B + O(\theta, \dot{\theta})$

$\dot{\phi} \rightarrow \dfrac{\partial \mathcal{L}}{\partial \dot{q}_2} = \phi$ $\quad \& \quad \dfrac{\partial \mathcal{L}}{\partial q_2} = -\theta B$

$\mathcal{L} = -\phi\dot{\theta} + \theta^2 B + O(\phi, \dot{\phi})$

$\dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \dot{\theta}$ $\qquad \dfrac{\partial \mathcal{L}}{\partial \phi} = -\dot{\theta} + 2\phi B$ $\qquad \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = -\dot{\phi}$

$\dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = -\dot{\phi}$ $\qquad \dfrac{\partial \mathcal{L}}{\partial \theta} = \dot{\phi} + 2\theta B$ $\rightarrow$

$2\dot{\theta} = 2\phi B \Rightarrow \dot{\theta} = \phi B$

$-2\dot{\phi} = 2\theta B \Rightarrow \dot{\phi} = -\theta B$