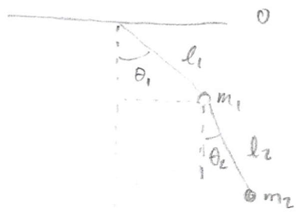
CM-2 January 2014

Coplanar double pendulum



$x_1 = l_1 \sin\theta_1$ $\quad y_1 = -l_1 \cos\theta_1$

$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$ $\quad y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$

$\dot{x}_1 = l_1 \cos\theta_1 \dot{\theta}_1$ $\quad \dot{y}_1 = l_1 \sin\theta_1 \dot{\theta}_1$

$\dot{x}_2 = l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2$

$\dot{y}_2 = l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2$

a. Find Lagrangian.

For m_1: $T_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 (l_1^2 \dot{\theta}_1^2)$

$V_1 = -mgh = -l_1 mg \cos\theta_1$

For m_2: $T_2 = \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) = \frac{1}{2} m_2 \left[l_1^2 \cos^2\theta_1 \dot{\theta}_1^2 + l_2^2 \cos^2\theta_2 \dot{\theta}_2^2 + 2 l_1 l_2 \cos\theta_1 \cos\theta_2 \dot{\theta}_1 \dot{\theta}_2 + \dots \right.$

$\left. \dots l_1^2 \sin^2\theta_1 \dot{\theta}_1^2 + l_2^2 \sin^2\theta_2 \dot{\theta}^2 + 2 l_1 l_2 \sin\theta_1 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2 \right]$

$= \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 [\underbrace{\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2}_{\cos(\theta_1 - \theta_2)}] \right]$

$V_2 = (-l_1 \cos\theta_1 - l_2 \cos\theta_2) mg$

$\mathcal{L} = T - U = \frac{1}{2}(m_1+m_2) l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2} l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)] + g(m_1+m_2) l_1 \cos\theta_1 + l_2 \cos\theta_2 m_2 g.$

6. Determine frequency of small oscilation

$l_1 = l_2 = l \qquad m_1 = m_2 = m$

$$\mathcal{L} = \underbrace{m l^2 \dot{\theta}_1^2 + \frac{1}{2} m l^2 \dot{\theta}_2^2}_{m l^2 (\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2)} + m l^2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + \underbrace{2mgl \cos\theta_1 + mgl\cos\theta_2}_{mgl(2\cos\theta_1 + \cos\theta_2)}$$

$$= ml^2 (\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2) + ml^2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) + mgl(2\cos\theta_1 + \cos\theta_2)$$

$\cos\theta_1 \approx 1 - \frac{\theta_1^2}{2} \qquad \cos\theta_2 \approx 1 - \frac{\theta_2^2}{2} \qquad \cos(\theta_1 - \theta_2) \approx 1 - \frac{(\theta_1 - \theta_2)^2}{2} \approx 1$

$$\mathcal{L} \approx ml^2 (\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2) + ml^2 \dot{\theta}_1 \dot{\theta}_2 + mgl\left(2 - \theta_1^2 + 1 - \frac{\theta_2^2}{2}\right)$$

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = \frac{\partial \mathcal{L}}{\partial \theta_1} \Rightarrow \frac{d}{dt}\left(2ml^2\dot{\theta}_1 + ml^2\dot{\theta}_2\right) = -2mgl\theta_1 \rightarrow 2ml^2\ddot{\theta}_1 + ml^2\ddot{\theta}_2 = -2mgl\theta_1$

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = \frac{\partial \mathcal{L}}{\partial \theta_2} \Rightarrow \frac{d}{dt}\left(ml^2\dot{\theta}_2 + ml^2\dot{\theta}_1\right) = -mgl\theta_2 \rightarrow ml^2\ddot{\theta}_2 + ml^2\ddot{\theta}_1 = -mgl\theta_2$

$M\ddot{\theta} + K\theta = 0$

$$\begin{bmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 2mgl & 0 \\ 0 & mgl \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow ml^2\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + mgl\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\det(K - \omega^2 M) = 0 \qquad \det\left|\frac{g}{l}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \omega^2\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right| = 0$

$$\left|\begin{bmatrix} 2\frac{g}{l} - 2\omega^2 & -\omega^2 \\ -\omega^2 & \frac{g}{l} - \omega^2 \end{bmatrix}\right| = 2\left(\frac{g}{l} - \omega^2\right)^2 - \omega^4 = 0 \Rightarrow 2\left[\frac{g^2}{l^2} - 2\frac{g}{l}\omega^2 + \omega^4\right] - \omega^4 = 0$$

$$2\frac{g^2}{l^2} - \left(4\frac{g}{l}\right)\omega^2 + \omega^4 = 0$$

$$\omega^2 = \frac{+\frac{4g}{l} \pm \sqrt{16\frac{g^2}{l^2} - 8\frac{g^2}{l^2}}}{2} = 2\frac{g}{l} \pm \frac{1}{2}\sqrt{8}\frac{g}{l} = 2\frac{g}{l} \pm \sqrt{2}\frac{g}{l}$$

$\omega^2 = (2 \pm \sqrt{2})\frac{g}{l} \qquad \omega = \pm\sqrt{\omega^2}$

$$\omega = \pm\sqrt{(2 \pm \sqrt{2})\frac{g}{l}}$$