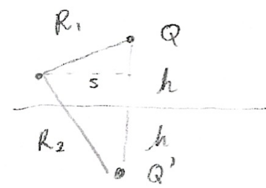
EM-1 January 2014

Maxwell equation: $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ and $\vec{\nabla} \times \vec{E} = 0$

$\Rightarrow \vec{E} = -\nabla \phi$

For $z>0$: $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_1}$



For $z<0$ $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_2} = 0$

Boundary conditions:

$$\varepsilon_1 E(z=0) = \varepsilon_2 E(z=0) \Rightarrow \begin{array}{l} \varepsilon_2 E_{2z} = \varepsilon_1 E_{1z} \\ E_{2x} = E_{1x} \\ E_{2y} = E_{1y} \end{array}$$

→ Method of images: place image charge Q'

Guess: put it at $z=-h$

$$\phi = \frac{1}{4\pi\varepsilon_1}\left(\frac{Q}{R_1} + \frac{Q'}{R_2}\right) \quad \to R_1 = \sqrt{s^2+(h-z)^2}, \quad R_2 = \sqrt{s^2+(h+z)^2} \quad \text{for } z>0$$

$$\phi = \frac{1}{4\pi\varepsilon_2}\frac{Q''}{R_1} \quad \text{for } z<0$$

→ Find Laplace equation with no singularity

⇒ Charge Q'' at position Q

Now to match boundary conditions:

$$\varepsilon_1 \frac{\partial \phi}{\partial z}\bigg|_{r=0} = \varepsilon_2 \frac{\partial \phi}{\partial z}\bigg|_{r=0}$$

$$\frac{\partial}{\partial z}\frac{1}{R_1} = \frac{\partial}{\partial z}\frac{1}{\sqrt{s^2+(h-z)^2}} = \frac{1}{2}\frac{2(h-z)}{(s^2+(h-z)^2)^{3/2}} = \frac{z-h}{(s^2+(h-z)^2)^{3/2}}$$

$$= \frac{-h}{(s^2+h^2)^{3/2}} = -\frac{\partial}{\partial z}\frac{1}{R_2}$$

$$(Q-Q') = (Q'')$$

$$\frac{\partial}{\partial s}\frac{1}{R_1} = \frac{-s}{(s^2+h^2)^{3/2}}$$

$$E_s = -\frac{\partial \Phi}{\partial s}$$

$$-\frac{s}{\varepsilon_1}(Q+Q') = -\frac{s}{\varepsilon_2}Q''$$

$$\frac{Q+Q'}{\varepsilon_1} = \frac{Q''}{\varepsilon_2}$$

I now will solve for Q' and Q''

$$\begin{cases} Q - Q' = Q'' \\ \dfrac{Q+Q'}{\varepsilon_1} = \dfrac{Q''}{\varepsilon_2} \end{cases}$$

$$\frac{Q+Q'}{\varepsilon_1} = \frac{Q-Q'}{\varepsilon_2} \rightarrow \frac{Q}{\varepsilon_1} - \frac{Q}{\varepsilon_2} = -\frac{Q'}{\varepsilon_2} - \frac{Q'}{\varepsilon_1}$$

$$Q\left[\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 \varepsilon_2}\right] = -Q'\left[\frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_2 \varepsilon_1}\right]$$

$$Q' = -Q\left[\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2}\right] = \underline{Q\left[\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right]} \rightarrow Q'$$

$$Q'' = Q - Q\left[\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right] = Q\left[\frac{\varepsilon_1 + \varepsilon_2 - \varepsilon_1 + \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right] = \underline{Q\left[\frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right]} \rightarrow Q''$$

Plugging back into the potentials:

$z > 0$

$$\Phi = \frac{1}{4\pi\varepsilon_1}\left\{\frac{Q}{\sqrt{s^2+(h-z)^2}} + \left[\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right]\frac{Q}{\sqrt{s^2+(h+z)^2}}\right\}$$

$z < 0$

$$\Phi = \frac{1}{2\pi} \frac{Q}{\sqrt{s^2+(h-z)^2}} \frac{1}{\varepsilon_1 + \varepsilon_2}$$

$$E = -\vec{\nabla}\Phi \qquad F = qE \qquad \rightarrow F_1 + F_2 = Q(E_1 + E_2)$$

$$E_2 = -\frac{\partial}{\partial z}\Phi - \frac{\partial}{\partial \theta}\Phi - \frac{\partial}{\partial s}\Phi$$

$$= \frac{-Q}{4\pi\varepsilon_1}\left\{\frac{1}{2}\frac{2(h-z)}{(s^2+(h-z)^2)^{3/2}} - \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\frac{h+z}{(s^2+(h+z)^2)^{3/2}}\right\} = \frac{Q}{4\pi\varepsilon_1}\left\{\left[\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right]\frac{(h+z)}{(s^2+(h+z)^2)^{3/2}} - \frac{h-z}{(s^2+(h-z)^2)^{3/2}}\right\}$$

$$E_1 = \frac{Q}{2\pi(\varepsilon_1 + \varepsilon_2)}\frac{(h-z)}{(s^2+(h-z)^2)^{3/2}} \rightarrow \text{no effect}$$

$$F_1 = \frac{Q^2}{2\pi}\left\{\frac{h-z}{(s^2+(h-z)^2)^{3/2}}\left[\frac{1}{\varepsilon_1+\varepsilon_2} - \frac{1}{2\varepsilon_1}\right] + \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\frac{h+z}{2(s^2+(h+z)^2)^{3/2}}\right)\right\}$$

$z = -h$
$s = 0$
(location of Q')

$$= \frac{Q^2}{2\pi}\frac{2h}{(4h^2)^{3/2}}\left[\frac{1}{\varepsilon_1+\varepsilon_2} - \frac{1}{2\varepsilon_1}\right] = \frac{Q^2}{8\pi h^2}\frac{2\varepsilon_1 - \varepsilon_1 - \varepsilon_2}{2\varepsilon_1(\varepsilon_1+\varepsilon_2)}$$

$$= \frac{Q^2}{16\pi h^2}\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1(\varepsilon_1+\varepsilon_2)}\hat{z} \rightarrow$$

if $\varepsilon_1 > \varepsilon_2$ → repulsive
$\varepsilon_2 < \varepsilon_1$ → attractive

b. Use these results to find force between Q in vacuum

$\varepsilon_1 \rightarrow \varepsilon_0$

infinite plate $\rightarrow \varepsilon_2 \rightarrow \infty$

$$\vec{F} = \frac{Q^2}{16\pi h^2} \frac{\varepsilon_0 - \varepsilon_2}{\varepsilon_0(\varepsilon_0 + \varepsilon_2)}$$

$$= \frac{Q^2}{16\pi\varepsilon_0 h^2} \left(\underbrace{\frac{\frac{\varepsilon_0}{\varepsilon_2} - 1}{\frac{\varepsilon_0}{\varepsilon_0} + 1}}_{\rightarrow \frac{-1}{1} = -1} \right)$$

$$= \frac{-Q^2}{16\pi\varepsilon_0 h^2}\hat{z} \rightarrow \text{attractive.}$$