CM-I January 2015

$H = (ap^2 + bq^2)^2$

a. Write down Hamilton's equations.

$\frac{\partial H}{\partial q} = -\dot{p}$ $\qquad\qquad$ $\frac{\partial H}{\partial p} = \dot{q}$

$$\frac{\partial H}{\partial q} = \frac{\partial}{\partial q}(ap^2+bq^2)^2 = 2(ap^2+bq^2)\cdot 2bq = 4bq(ap^2+bq^2) = -\dot{p}$$

$$\frac{\partial H}{\partial p} = \frac{\partial}{\partial p}(ap^2+bq^2)^2 = 2(ap^2+bq^2)\cdot 2ap = 4ap(ap^2+bq^2) = \dot{q}$$

b. Solve these equations.

$p(t=0) = p_0$

$q(t=0) = q_0$

$$\dot{p} = -4bq(ap^2+bq^2) = -4abq(p^2) - 4b^2q^3$$

$$\dot{q} = 4ap(ap^2+bq^2) = 4a^2p^3 + 4abp(q^2)$$

$\frac{-\dot{p}}{4bq} = \frac{\dot{q}}{4ap}$ $\qquad$ $\dot{p} = -\frac{bq\dot{q}}{ap}$ $\qquad$ $p\dot{p} = -\frac{b}{a}q\dot{q}$

$$\frac{d}{dt}\dot{p} = \frac{d}{dt}\left[-4bq(ap^2+bq^2)\right] \Rightarrow \ddot{p} = -4b\dot{q}(ap^2+bq^2) - 4bq(2ap\dot{p} + \underbrace{2bq\dot{q}}_{-\frac{a}{b}p\dot{p}})$$

$$= -4b\dot{q}(ap^2+bq^2)$$

Similarly.

$$\ddot{q} = 4a\dot{p}(ap^2+bq^2)$$

$\frac{\ddot{p}}{\ddot{q}} = \frac{-b}{a}\frac{\dot{q}}{\dot{p}}$ & $\frac{\dot{p}}{\dot{q}} = -\frac{b}{a}\frac{q}{p}$ $\Rightarrow$ $\frac{\ddot{p}}{\ddot{q}} = \frac{p}{q}$ $\qquad$ $\frac{\ddot{p}}{p} = \frac{\ddot{q}}{q} = -\omega^2$

$p(t) = A\sin(\omega t) + p_0\cos(\omega t)$ & $q(t) = B\sin(\omega t) + q_0\cos(\omega t)$

$$p\dot{p} = -\frac{b}{a}q\dot{q} \Rightarrow \left[A\sin(\omega t) + p_0\cos(\omega t)\right]\left[A\omega\cos(\omega t) - p_0\sin(\omega t)\omega\right] = -\frac{b}{a}\left[B\sin(\omega t) + q_0\cos(\omega t)\right]\left[B\omega\cos(\omega t) - q_0\sin(\omega t)\omega\right]$$

$$p_0A\cos^2(\omega t) - p_0A\sin^2(\omega t) - p_0^2\cos(\omega t)\sin(\omega t) = -\frac{b}{a}\left[q_0B\cos^2(\omega t) - q_0B\sin^2(\omega t) - q^2\cos(\omega t)\sin(\omega t)\right]$$

At $t=0$ $\rightarrow$

$$p_0A\cos^2(\omega t) = -\frac{b}{a}q_0B\cos^2(\omega t)$$

$$Ap_0 = -\frac{b}{a}q_0B$$

$$A = -\frac{b}{a}\frac{q_0}{p_0}B$$

Now to find B:

$$p(t) = \frac{-b}{a}\frac{q_0}{p_0} B\sin(wt) + p_0 \cos(wt)$$

$$q(t) = B\sin(wt) + q_0 \cos(wt)$$

$$\dot{p} = -4bq(ap^2 + bq^2)$$

$$-\frac{b}{a}\frac{q_0}{p_0} Bw\cos(wt) - p_0 w \sin(wt) = -4b\left[B\sin(wt) + q_0\cos(wt)\right]\left\{a\left[\frac{-bq_0}{ap_0}B\sin(wt) + p_0\cos(wt)\right]^2 + b\left[B\sin(wt) + q_0\cos(wt)\right]^2\right\}$$

$$\dot{q} = 4ap(ap^2 + bq^2)$$

$$Bw\cos(wt) - wq_0\sin(wt) = 4a\left[-\frac{b}{a}\frac{q_0}{p_0}B\sin(wt) + p_0\cos(wt)\right]\left\{a\left[\frac{-bq_0}{ap_0}B\sin(wt) + p_0\cos(wt)\right]^2 + b\left[B\sin(wt) + q_0\cos(wt)\right]^2\right\}$$

$t=0$ from $\dot{q}$

$$Bw = (4ap_0)(ap_0^2 + bq_0^2)$$

$t=0$ from $\dot{p}$

$$-\frac{b}{a}\frac{q_0}{p_0}bw = -4bq_0(ap_0^2 + bq_0^2)$$

$$w = \frac{4bq_0}{bq_0B}\cdot ap_0(ap_0^2 + bq_0^2)$$

$$w = \frac{4ap_0}{B}(ap_0^2 + bq_0^2)$$

$$p\dot{p} = -\frac{b}{a}q\dot{q}$$

$$\left[A\sin(wt) + p_0\cos(wt)\right]\left[Aw\cos(wt) - p_0 w\sin(wt)\right] = -\frac{b}{a}\left[B\sin(wt) + q_0\cos(wt)\right]\left[Bw\cos(wt) - wq_0\sin(wt)\right]$$

$$A^2\sin(wt)\cos(wt) + \underbrace{Ap_0\cos^2(wt) - Ap_0\sin^2(wt)}_{Ap_0\cos(2wt)} - p_0^2\cos(wt)\sin(wt) = -\frac{b}{a}\left[B^2\sin\cos + \underbrace{q_0B\cos^2 - q_0^2\cos\sin - Bq_0\sin^2(wt)}_{Bq_0\cos(2wt)}\right]$$

→ I simplified this online to solve for B

$$B = -\sqrt{\frac{a}{b}}\,p_0 \quad \Rightarrow \quad A = -\frac{b}{a}\frac{q_0}{p_0}\left(\frac{\sqrt{a}}{\sqrt{b}}p_0\right) = \sqrt{\frac{b}{a}}\,q_0$$

$$w = \frac{-4ap_0}{p_0\sqrt{\frac{b}{a}}}(ap_0^2 + bq_0^2) \Rightarrow w = -4\sqrt{ab}\,(ap_0^2 + bq_0^2)$$

$$\Rightarrow p(t) = \sqrt{\frac{b}{a}}\,q_0\sin\left[-4\sqrt{ab}\,(ap_0^2 + bq_0^2)t\right] + p_0\cos\left[-4\sqrt{ab}\,(ap_0^2 + bq_0^2)t\right]$$

$$q(t) = -\sqrt{\frac{a}{b}}\,p_0\sin\left[-4\sqrt{ab}\,(ap_0^2 + bq_0^2)t\right] + q_0\cos\left[-4\sqrt{ab}\,(ap_0^2 + bq_0^2)t\right]$$