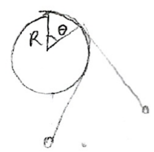
CM-1 January 2017

Pendulum at top of fixed disk

$\downarrow g$



mass of bob: m

$l > \pi R$ $\qquad$ l is fixed

Constraint: θ

a. Find equations of motion

$\mathcal{L} = T - U$ $\qquad$ $l' = l - (\theta \cdot R)$

$V = \omega R$

$U = mgh$ $\qquad$ $T = \frac{1}{2}mv^2$ $\qquad$ $v = \dot{\theta} \cdot (l - \theta R)$

$h = R\cos\theta - [l - \theta R]\sin\theta$

$$\mathcal{L} = \frac{1}{2}m\dot{\theta}^2(l - \theta R)^2 - mg\left(R\cos\theta - [l - \theta R]\sin\theta\right)$$

Now:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta}$$

$$\frac{d}{dt}\left(m\dot{\theta}(l - \theta R)^2\right) = -m\dot{\theta}^2(l - \theta R)R - mg\left(-R\sin\theta + R\sin\theta + [l - \theta R]\cos\theta\right)$$

$$m\ddot{\theta}(l - \theta R)^2 - 2m\dot{\theta}^2(l - \theta R)R = -m\dot{\theta}^2(l - \theta R)R + mg\,[l - \theta R]\cos\theta$$

$$\ddot{\theta}(l - \theta R)^2 = R\dot{\theta}^2(l - \theta R) + g\,[l - \theta R]\cos\theta$$

$$\ddot{\theta} = \frac{1}{l - \theta R}\left[R\dot{\theta}^2 + g\cos\theta\right]$$

b. What is the equilibrium angle θ_0?

$0 = \frac{1}{l - \theta R}\, g\cos\theta$ $\quad \Rightarrow \cos\theta = 0$ for $\theta = \frac{\pi}{2}$, $= \frac{3\pi}{4}$

$\theta = \frac{\pi}{2}$ $\quad \rightarrow$ stable equilibrium

$\theta = \frac{3\pi}{4}$ $\quad \rightarrow$ unstable equilibrium $\frac{3\pi}{4} \rightarrow$ why not 0

c. Find frequency of oscillation.

$$\ddot{\theta} = \frac{1}{l - \theta R}\left(R\dot{\theta}^2 + g\cos\theta\right)$$

Variational principle:

$\theta = \theta_0 + \eta$ for small displacement η about equilibrium θ_0

$$\ddot{\eta} = \frac{1}{l-(\theta_0+\eta)R}\left[R\dot{\eta}^2 + g\cos(\theta_0+\eta)\right]$$

$$\cos(a+b) = \underbrace{\cos a\cos b}_{\cos\theta_0 \cdot 1} - \underbrace{\sin a\sin b}_{\eta\sin\theta_0}$$

$$\ddot{\eta}(l-[\theta_0+\eta]R) = R\dot{\eta}^2 + g\cos(\theta_0+\eta)$$

$$\ddot{\eta}(l-\theta_0 R) + \ddot{\eta}\eta R = R\dot{\eta}^2 + g(\cos\theta_0 + \eta\sin\theta_0)$$

terms of order $\eta^2 \to 0$ since $\eta \ll \theta_0$

$$\ddot{\eta}(l-\theta_0 R) = g(\cos\theta_0 + \eta\sin\theta_0)$$

$$\theta = \ddot{\eta} - \frac{g\cos\theta_0}{l-\theta_0 R} - \underbrace{\frac{g\sin\theta_0}{l-\theta_0 R}}_{w^2}\eta$$

$f = \frac{1}{T}$

$$w = \sqrt{\frac{g\sin\theta_0}{l-\theta_0 R}}$$

Differential equations

$$\frac{d^2\theta}{dt^2} + w^2\theta = 0$$

$$\theta(t) = \theta_0\cos(wt)$$