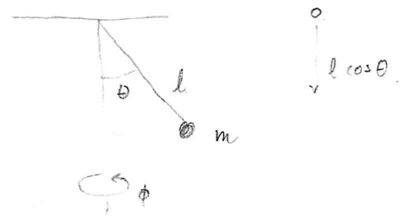
CM-1 September 2013

Spherical Pendulum



a. Write Lagrangian and equations of motion

$x = l\sin\theta\cos\phi$

$y = l\sin\theta\sin\phi$

$z = l\cos\theta$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right)l^2$$

$$U = mg(1-\cos\theta)\,l$$

$$\mathcal{L} = T - U = \frac{1}{2}ml^2\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) - mgl(1-\cos\theta)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = \frac{\partial\mathcal{L}}{\partial\theta} \Rightarrow \frac{d}{dt}\left(ml^2\dot\theta\right) = ml^2\sin\theta\cos\theta\,\dot\phi^2 - mgl\sin\theta$$

$$\ddot\theta = \sin\theta\cos\theta\,\dot\phi^2 - \frac{g}{l}\sin\theta$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\phi} = \frac{\partial\mathcal{L}}{\partial\phi} \Rightarrow \frac{d}{dt}\left(ml^2\sin^2\theta\,\dot\phi\right) = 0 \rightarrow ml^2\left(2\sin\theta\cos\theta\,\dot\theta\dot\phi + \sin^2\theta\,\ddot\phi\right) = 0$$

$$\ddot\phi = -\frac{2\cos\theta}{\sin\theta}\dot\theta\dot\phi$$

b. Write Hamiltonian & Hamiltonian equations of motion.

$$p_\theta = \frac{\partial\mathcal{L}}{\partial\dot\theta} = ml^2\dot\theta$$

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = ml^2\sin^2\theta\,\dot\phi$$

$$H = p_\theta\dot\theta + p_\phi\dot\phi - \mathcal{L}$$

$$= ml^2\dot\theta^2 + ml^2\sin^2\theta\,\dot\phi^2 - \frac{1}{2}ml^2\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) + mgl(1-\cos\theta)$$

$$= \frac{1}{2}ml^2\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) + mgl(1-\cos\theta)$$

$$= \frac{1}{2ml^2}\left[p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right] + mgl(1-\cos\theta)$$

$$\frac{\partial H}{\partial q_i} = -\dot p_i \Rightarrow \frac{\partial H}{\partial\theta} = \frac{-2}{2ml^2}p_\phi^2\cdot\sin^{-3}\theta\cdot\cos\theta + mgl\sin\theta \rightarrow \dot p_\theta = \frac{\cos\theta}{\sin^3\theta}\frac{p_\phi^2}{ml^2} - mgl\sin\theta$$

$$\Rightarrow \frac{\partial H}{\partial\phi} = 0 \rightarrow \dot p_\phi = 0$$

$$\frac{\partial H}{\partial p_i} = \dot q_i \Rightarrow \frac{\partial H}{\partial p_\theta} = \frac{1}{ml^2}p_\theta \rightarrow \dot\theta = \frac{p_\theta}{ml^2}$$

$$\Rightarrow \frac{\partial H}{\partial p_\phi} = \frac{1}{ml^2}\frac{p_\phi}{\sin^2\theta} \rightarrow \dot\phi = \frac{p_\phi}{ml^2\sin^2\theta}$$

c. Show they are equivalent and discuss angular momentum

$$\ddot{\theta} = \sin\theta\cos\theta\,\dot{\phi}^2 - \frac{g}{l}\sin\theta$$

$$\ddot{\phi} = \frac{-2\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi}$$

$$\dot{P}_\phi = 0$$

$$\dot{P}_\theta = \frac{\cos\theta}{\sin^3\theta}\frac{P_\phi^2}{ml^2} - mgl\sin\theta$$

$$\dot{\theta} = \frac{P_\theta}{ml^2}$$

$$\dot{\phi} = \frac{P_\phi}{ml^2\sin^2\theta}$$

Start from Hamilton's Equations:

$$\ddot{\phi} = \frac{-2P_\phi\cos\theta}{ml^2\sin^3\theta}\dot{\theta}$$

Plug $P_\phi = ml^2\sin^2\theta\,\dot{\phi}$

$$= \frac{-2\cos\theta}{ml^2\sin^3\theta}\dot{\theta}\left(ml^2\sin^2\theta\,\dot{\phi}\right) = -\frac{2\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi}$$

→ same as Lagrange equation.

$$\ddot{\theta} = \frac{\dot{P}_\theta}{ml^2} = \frac{1}{ml^2}\left(\frac{\cos\theta}{\sin^3\theta}\frac{P_\phi^2}{ml^2} - mgl\sin\theta\right) = \frac{1}{ml^2}\left(\frac{\cos\theta}{\sin^3\theta}\frac{m^2l^4\sin^4\theta\,\dot{\phi}^2}{ml^2} - mgl\sin\theta\right)$$

$$= \cos\theta\sin\theta\,\dot{\phi}^2 - \frac{g}{l}\sin\theta$$

→ same as Lagrange equation

→ We can see that P_ϕ is a constant but P_θ is not, so angular momentum in ϕ is conserved but not in θ.

d. Discuss solution for small amplitude.

$\theta = \theta_0 + \delta\theta$

$$\delta\ddot{\theta} = \sin(\theta_0+\delta\theta)\cos(\theta_0+\delta\theta)\,\dot{\phi}^2 - \frac{g}{l}\sin(\theta_0+\delta\theta)$$

$$\ddot{\phi} = \frac{-2\cos(\theta_0+\delta\theta)}{\sin(\theta_0+\delta\theta)}\delta\dot{\theta}\,\dot{\phi}$$

Take $\sin x = x$

$\cos x = 1 - \frac{x^2}{2}$

$$\dot{\phi}^2 = \left(\frac{-\sin\theta}{2\cos\theta}\frac{\ddot{\phi}}{\dot{\theta}}\right)^2$$

$$\delta\ddot{\theta} = \frac{1}{2}\sin(2(\theta_0+\delta\theta))\,\dot{\phi}^2 - \frac{g}{l}(\theta_0+\delta\theta)$$

P_ϕ = constant → $ml^2\sin^2\theta\,\dot{\phi} = L_0$

$$\approx (\theta_0+\delta\theta)\,\dot{\phi}^2 - \frac{g}{l}(\theta_0+\delta\theta) = \delta\theta\left(\dot{\phi}^2 - g/l\right) + K$$

$K = \theta_0(\dot{\phi}^2 - g/l)$

→ this will give a small oscillation about θ_0. (unsure haha)