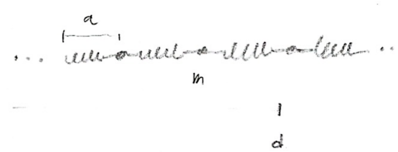
CM-1 September 2018

Infinite chain - mass m

spring constant β

uniform spacing a



a. Write down equations of motion.

$d \to$ equilibrium location of mass

$d_i = ia$

$T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2 \Rightarrow \sum_i \frac{1}{2} m \dot{x}_i^2$

$V = \frac{1}{2}\beta (\Delta x)^2 \Rightarrow \sum_i \frac{1}{2}\beta \left[(x_i - x_{i-1})^2 + (x_{i+1} - x_i)^2\right]$

$x_i^2 - 2x_i x_{i-1} + x_{i-1}^2 + x_{i+1}^2 - 2x_{i+1}x_i + x_i^2$

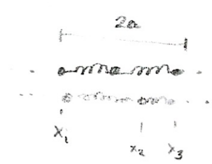
$$\mathcal{L} = \frac{1}{2}\sum_i m\dot{x}_i^2 + \beta\left[(x_i - x_{i-1})^2 + (x_{i+1} - x_i)^2\right]$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = \frac{d}{dt} m\dot{x}_i = m\ddot{x}_i$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\beta}{2}\left[\frac{\partial}{\partial x_i}(x_i - x_{i-1})^2 + \frac{\partial}{\partial x_i}(x_{i+1} - x_i)^2\right] = \beta\left[(x_i - x_{i-1}) - (x_{i+1} - x_i)\right] = \beta\left[-x_{i-1} + 2x_i - x_{i+1}\right]$$

$$\Rightarrow \quad m\ddot{x}_i = \beta(-x_{i-1} + 2x_i - x_{i-1})$$

$$m\ddot{x}_i = \beta(-x_{i-1} + 2x_i - x_{i-1}) \Rightarrow \ddot{x}_i = \frac{\beta}{m}(-x_{i-1} + 2x_i - x_{i-1})$$



$x_1 + x_2 + x_3$

b) Find frequency vs. wavelength for all possible normal modes.

ω vs. q

Normal modes: $x_j = A_j \cos(\omega t + \phi)$.

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_j \\ A_{j+1} \\ \vdots \end{bmatrix}$$

$\ddot{X} = -BX$

For $$B = \begin{bmatrix} & & \vdots & & & \\ & -\frac{2\beta}{m} & \frac{\beta}{m} & 0 & 0 & \\ \cdots & \frac{\beta}{m} & -\frac{2\beta}{m} & \frac{\beta}{m} & 0 & \cdots \\ & 0 & \frac{\beta}{m} & -\frac{2\beta}{m} & \frac{\beta}{m} & \\ & & \vdots & & & \end{bmatrix}$$

→ we can translate all system by a, since it is infinite (left most column → 0)

$$\tilde{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & \\ \vdots & & & & \end{bmatrix}$$ → find eigenvalues of $\tilde{B}$

$BA = -\frac{\beta}{m} A_{j-1} + \frac{2\beta}{m} A_j - \frac{\beta}{m} A_{j+1}$

$\tilde{B}A = cA \Rightarrow \tilde{B}A = \begin{bmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \\ & & \vdots & & \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} A_2 \\ A_3 \\ A_4 \\ \vdots \end{bmatrix} \Rightarrow \tilde{B}A = \begin{bmatrix} A_2 \\ A_3 \\ \vdots \\ A_{j+1} \\ A_{j+2} \\ \vdots \end{bmatrix}$

$\Rightarrow cA_i = A_{i+1}$

Say $A_0 = 1$

$\Rightarrow$ $i=0$ $\quad cA_0 = A_1$

$i=1$ $\quad cA_1 = A_2 = c^2$

$\rightarrow A_0 = 1, \; A_1 = c, \; A_2 = c^2 \; \ldots \; A_i = c^i$

$BA = \omega^2 A$ for harmonic oscillator. and $\omega_0^2 = \frac{\beta}{m}$ | $\tilde{B}A = \omega^2 A \rightarrow \omega^2 A_i = \omega^2 c^i$

$-\frac{\beta}{m}(A_{j-1} + 2A_j - A_{j+1}) = \omega_0^2(-A_{j-1} + 2A_j - A_{j+1})$

$A_j = c^j$

$$= \underbrace{\omega_0^2\left(-\frac{1}{c} + 2 - c\right)}_{\omega^2}$$

$\omega^2 = \omega_0^2\left(\frac{-1}{c} + 2 - c\right)$ → for $c = e^{ika}$

$\omega^2 = \omega_0^2\left(-e^{ika} + 2\, e^{ika}\right) = \omega_0^2(2 - 2\cos(ka)) = \omega_0^2\left(2 - 2\cos\left(\frac{2\pi a}{q}\right)\right)$

q - wavelength.