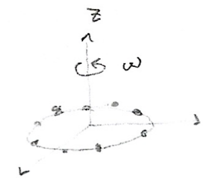
September 2020

Disk radius R
mass M
moment of inertia I

n point charges (charge q) $\to$ around rim.

Immersed in $\vec{B}(r,z) = K(-r\hat{r} + 2z\hat{z})$

$t=0 \to$ rotates with ω_0



a. Find net force on disk.

$\vec{F} = q\vec{v}\times\vec{B}$ $\qquad\qquad \vec{v} = R\omega\hat{\phi} + v_z\hat{z}$

$= q(R\omega\hat{\varphi} + v_z\hat{z}) \times K(-R\hat{r} + 2z\hat{z})$

$$= q\begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ 0 & \omega R & v_z \\ -RK & 0 & 2zK \end{vmatrix} = q\left[\hat{r}\ \omega R(2zK) - \hat{\phi}\ RKv_z + \hat{z}\ K\omega R^2\right]$$

$\hat{\phi} \to$ cancel out
$\hat{r} \to$ cancel out

$\vec{F}_T = \sum_i^n \vec{F}_i = nqK\omega R^2\hat{z}$

b. Find net torque.

$$\tau_i = \vec{r}_i \times \vec{F}_i = \begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ R & 0 & 0 \\ 2\omega zR & Rv_z & \omega R^2 \end{vmatrix} qK = -qK\left[\hat{\phi}\ \omega R^3 + \hat{z}\ R^2 v_z\right]$$

$\hat{\phi}$ will cancel out

$\tau_T = \sum_i^n \tau_i = -nqKR^2v_z\hat{z}$

c. Find $\omega(t)$

$\tau = I\times\alpha \quad\Rightarrow\quad -nqKR^2v_z = I\alpha = I\frac{d\omega}{dt}$

$$\frac{d\omega}{dt} = \frac{-nqKR^2v_z}{I} = \frac{-nqKR^2}{I}\frac{dz}{dt}$$

$F_z = m\frac{d^2z}{dt^2} = nqK\omega R^2 = M\ddot{z} \quad\to\quad \ddot{z} = \frac{nqK\omega R^2}{M}$

$$\frac{d^2\omega}{dt^2} = \frac{-nqKR^2}{I}\frac{d^2z}{dt^2} = \frac{-nqKR^2}{I}\left(\frac{nqK\omega R^2}{M}\right) = -\frac{(nqKR^2)^2}{IM}\omega$$

$\frac{d^2\omega}{dt^2} = -\alpha^2\omega$

$\alpha = \frac{nqKR^2}{\sqrt{IM}}$

$$\omega(t) = \omega_0\cos\left(\frac{nqKR^2}{\sqrt{IM}}t\right)$$

d. Find position of disk $z(t)$

$$F_z = m a_z = M\ddot{z} \Rightarrow \ddot{z} = \left(\frac{nqKR^2}{M}\right)\omega = \frac{nqKR^2}{M}$$

$$\frac{dz}{dt} = \frac{-I}{nqKR^2}\frac{d\omega}{dt} = \frac{-I}{nqKR^2}\left[-\omega_0 \sin\left(\frac{nqKR^2}{\sqrt{IM}}t\right)\cdot\frac{nqKR^2}{\sqrt{IM}}\right] = \sqrt{\frac{I}{M}}\,\omega_0\left[\sin\left(\frac{nqKR^2}{\sqrt{IM}}t\right)\right]$$

$$z(t) = \int_0^{z(t)} \sqrt{\frac{I}{M}}\,\omega_0\left[\sin\left(\frac{nqKR^2}{\sqrt{IM}}t\right)\right]dt = \sqrt{\frac{I}{M}}\,\omega_0\int_0^t \sin(\alpha t)\,dt = \frac{\omega_0\sqrt{IM}}{nqKR^2}\sqrt{\frac{I}{M}}\left[1-\cos\left(\frac{nqKR^2}{\sqrt{IM}}\right)\right]$$

$$z(t) = \frac{\omega_0 I}{nqKR^2}\left[1-\cos\left(\frac{nqKR^2}{\sqrt{IM}}\right)\right]$$

e. Describe motion and check kinetic energy.

$$T = \frac{1}{2}mv_z^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}M\left[\frac{I}{M}\omega_0^2\sin^2(\alpha t)\right] + \frac{1}{2}I\left[\omega_0^2\cos^2(\alpha t)\right]$$

$$= \frac{1}{2}I\omega_0^2\sin^2(\alpha t) + \frac{1}{2}I\omega_0^2\cos^2(\alpha t) = \frac{1}{2}I\omega_0^2\left[\sin^2(\alpha t)+\cos^2(\alpha t)\right] = \frac{1}{2}I\omega_0^2$$

$\rightarrow$ kinetic energy remains constant.

Ring goes up and down, at top and bottom it is spinning the fastest and halfway up it is not spinning. It starts by moving in z.