SM-2 January 2014

N - classical harmonic oscillators on lattice $\rightarrow$ distinguishable.

ν - natural frequency

Use microcanonical ensemble.

Find entropy as function of N, ν and T

$$S(E,N) = k_B \ln \Omega$$

$$H_x = \frac{p_x^2}{2m} + \frac{1}{2} m \nu^2 x^2$$

$$H = \sum_{i=1}^{3N} \left(\frac{p_i^2}{2m} + \frac{m\nu^2 q_i^2}{2} \right)$$

Define $\tilde{q}_i \equiv \sqrt{m\omega}\, q_i$ and $\tilde{p}_i \equiv \frac{p_i}{\sqrt{m\omega}}$

$$\Rightarrow \tilde{E} = \frac{\omega}{2} \sum_{i=1}^{3N} \left(\tilde{p}_i^{\,2} + \tilde{q}_i^{\,2} \right)$$

$$\Omega \approx \frac{1}{h^3} \int_{H \le E} d\tilde{q}_1 - d\tilde{q}_{3N}\, d\tilde{p}_1 - dp_{3N} \qquad R = \sqrt{\frac{2E}{\omega}} \qquad R^{6N} = \left(\frac{2E}{\nu}\right)^{3N}$$

Volume of unit sphere: $\dfrac{\pi^{6N/2}}{\Gamma\left(\frac{6N}{2}+1\right)} = \dfrac{\pi^{3N}}{\Gamma(3N+1)} = \dfrac{\pi^{3N}}{3N!}$

$$\Omega \approx \frac{\pi^{3N} \left(\frac{2E}{\nu}\right)^{3N}}{(3N)!\; h^{3N}} = \frac{\pi^{3N}}{(3N)!\, h^{3N}} \left(\frac{2E}{\nu}\right)^{3N} = \left(\frac{2E\pi}{h\nu}\right)^{3N} \frac{1}{(3N)!}$$

$$S = k_B \ln \Omega = k_B \ln\left[\left(\frac{2E\pi}{h\nu}\right)^{3N} \left(\frac{1}{3N!}\right)\right] = k_B \left[3N \ln \frac{2E\pi}{h\nu} - \underbrace{\ln(3N!)}_{3N \log(3N) - 3N} \right]$$

$$= \left\{ 3N \left[\ln \frac{2E\pi}{h\nu} - \ln(3N) - 1 \right] \right\} k_B \approx k_B 3N \ln\left(\frac{2}{3} \frac{E\pi}{h\nu N} \right)$$

$$\frac{\partial S}{\partial E} = \frac{1}{T} \Rightarrow 3Nk_B \frac{3}{2}\frac{h\nu N}{E\pi} \cdot \frac{2}{3}\frac{\pi}{h\nu N} = \frac{3Nk_B}{E} = T$$

$$E = 3Nk_B T \qquad \rightarrow \qquad S = 3Nk_B \ln\left(\frac{2}{3} \frac{3Nk_B T\pi}{h\nu N} \right)$$

$$= 3Nk_B \ln\left(\frac{2k_B T\pi}{h\nu} \right)$$