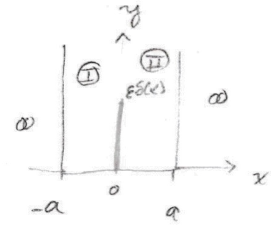
QM-1 January 2015

Particle in box width 2a
δ function at the origin

$$V(x) = \begin{cases} \infty & \text{for } |x| \geq a \\ \varepsilon \delta(x) & \text{for } |x| < a \end{cases}$$



a. Consider even eigenfunctions of x in terms of k.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

Conditions:
$\psi = 0$ at $x = a$ and $x = -a$
$\psi_I = \psi_{II}$ at $x = 0$

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{d^2\psi}{dx^2}dx + \int_{-\varepsilon}^{\varepsilon}V(x)\psi\, dx = \underbrace{\int_{-\varepsilon}^{\varepsilon}E\psi}_{=0}$$

$= 0 \rightarrow$ it is continuous.

$$\frac{d}{dx}\psi_{II}(0) - \frac{d}{dx}\psi_I(0) \qquad \varepsilon\psi(0)$$

$$-\frac{\hbar^2}{2m}\left[\frac{d}{dx}\psi_{II}(0) - \frac{d}{dx}\psi_I(0)\right] = -\varepsilon\psi(0) \Rightarrow \frac{d}{dx}\psi_{II}(0) - \frac{d}{dx}\psi_I(0) = \frac{2m\varepsilon}{\hbar^2}\psi(0)$$

$$\psi_I = A\cos(kx) + B\sin(kx)$$
$$\psi_{II} = C\cos(kx) + D\sin(kx)$$

$\rightarrow$ even function: $f(x) = f(-x)$
$\rightarrow$ cos will be even, so to make sin even $(A=C)$

$$B\sin(kx) = D\sin(-kx)$$
$$B = -D$$

$$\Rightarrow \psi_I = A\cos(kx) + B\sin(kx)$$
$$\psi_{II} = A\cos(kx) - B\sin(kx)$$

at 0 $\Rightarrow \psi(0) = A = \psi_I(0) = \psi_{II}(0)$ ✓

$$\frac{d}{dx}\psi_I = -Ak\sin(kx) + Bk\cos(kx)$$
$$\frac{d}{dx}\psi_{II} = -Ak\sin(kx) - Bk\cos(kx)$$

$$\Rightarrow \frac{d}{dx}\psi_{II}(0) - \frac{d}{dx}\psi_I(0) = -Bk\cos(kx) - Bk\cos(kx) = -2Bk\cos(kx)$$

$$\Rightarrow -2Bk\cos(kx)\Big|_0 = \frac{2m\varepsilon}{\hbar^2}\psi(0) = \frac{2m\varepsilon}{\hbar^2}A$$

$$-2Bk = \frac{2m\varepsilon}{\hbar^2}A \Rightarrow B = \frac{-m\varepsilon}{k\hbar^2}A$$

To solve for B in terms of A:

$\psi_I(-a) = \psi_{II}(a) = 0$

$A\cos(ka) - B\sin(ka) = 0$

$A\cos(ka) = B\sin(ka)$

$A = B\tan(ka)$

$\Rightarrow \quad \psi_I = B\tan(ka)\cos(kx) + B\sin(kx)$

$\psi_{II} = B\tan(ka)\cos(kx) - B\sin(kx)$

b. Find equation for k that determine energy eigenvalues.

→ from bottom of last page:

$$B = \frac{-m\varepsilon}{\hbar^2 k} A = \frac{-m\varepsilon}{\hbar^2 k} B\tan(ka)$$

$$\tan(ka) = \frac{-\hbar^2 k}{m\varepsilon} \qquad \text{for } k^2 = \frac{2mE}{\hbar^2}$$

c. Solve b when $\hbar^2/ma\varepsilon \ll 1$. Find energy levels to leading order in $\hbar^2/ma\varepsilon$

$z = ka$

$$\tan(z) = \frac{-\hbar^2 z}{m\varepsilon a} \qquad z_0 = \frac{-m\varepsilon a}{\hbar^2}$$

$$= \frac{z}{z_0} \qquad z_0 \gg 1 \Rightarrow \frac{z}{z_0} \to 0$$

$\tan(z) \to 0$

$\tan(ka) \to 0 \qquad ka = n\pi$ and $\tan(ka) \to ka$ up to leading order

$$n\pi = \frac{-\hbar^2\sqrt{2mE}}{m\varepsilon\hbar} \qquad -\frac{nm\varepsilon\pi}{\hbar} = \sqrt{2mE}$$

$$E = \frac{n^2 m\,\varepsilon^3\pi^2}{2\hbar^2} \quad \text{for } n = 0, 1, 2\ldots$$

d. Now consider odd eigenfunctions: $-f(x) = f(-x)$

$\psi_I = A\cos(kx) + B\sin(kx) \qquad \psi_{II} = C\cos(kx) + D\sin(kx)$

$\rightarrow A\cos(-kx) + B\sin(-kx) = -C\cos(kx) - D\sin(kx)$

$A = -C$ and $B = D$ $\qquad$ at $x = 0$. $-A = A \Rightarrow A = 0$

$\psi_I = A\cos(kx) + B\sin(kx)$

$\psi_{II} = -A\cos(kx) + B\sin(kx) \quad \Rightarrow \quad \psi_I = B\sin(kx)$

$\psi_{II} = B\sin(kx)$

→ continued

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Energy eigenvalues

$\psi_{I}(a) = \psi_{II}(a) = 0$ $\qquad B \sin(ak) = 0$

$ak = \pi n \quad$ for $n = 0, 1, 2 \ldots$

$a \frac{\sqrt{2mE}}{\hbar} = n\pi \implies \frac{n\pi\hbar}{a} = \sqrt{2mE} \qquad E = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$ for $n = 0, 1, 2 \ldots$

→ The delta function does not play a role as odd wave functions have $\psi(0) = 0$ so they are unaffected by the potential at this location.