QM-3 January 2015

3D Harmonic oscillator

$$\hat{H} = \frac{|\vec{p}|^2}{2m} + \frac{1}{2} m\omega^2 |\vec{x}|^2$$

a. What are the energies and degeneracies of three lowest levels?

$$H_x = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2 \qquad H_y = \frac{p_y^2}{2m} + \frac{1}{2} m\omega^2 y^2 \qquad H_z = \frac{p_z^2}{2m} + \frac{1}{2} m\omega^2 z^2$$

$$E_x = \hbar\omega\left(n_x + \frac{1}{2}\right) \qquad E_y = \hbar\omega\left(n_y + \frac{1}{2}\right) \qquad E_z = \hbar\omega\left(n_z + \frac{1}{2}\right)$$

Three lowest states

0: $E = \frac{3}{2}\hbar\omega$ $\quad |000\rangle$ – 0 degeneracy

1: $E = \frac{5}{2}\hbar\omega$ $\quad |100\rangle\ |010\rangle\ |001\rangle$ – 3 degeneracy

2: $E = \frac{7}{2}\hbar\omega$ $\quad |110\rangle\ |101\rangle\ |011\rangle\ |200\rangle\ |020\rangle\ |002\rangle$ – 6 degeneracy

b. Classify these levels into states of fixed angular momentum.

$n=0$ $\quad |000\rangle$ $\quad m = 0$

$n=1$ $\quad |100\rangle$ $\quad m = -1$ $\qquad |010\rangle$ $\quad m = 0$ $\qquad |001\rangle$ $\quad m = 1$ $\qquad l = 1$

$n=2$ $\quad |110\rangle$ $\quad m = -1$ $\qquad |101\rangle$ $\quad m = 0$ $\qquad |011\rangle$ $\quad m = 1$ $\qquad l = 1$

$\quad |200\rangle$ $\quad m = -2$ $\qquad |020\rangle$ $\quad m = 0$ $\qquad |002\rangle$ $\quad m = 2$ $\qquad l = 2$

c. Calculate change in ground state energy due to perturbation. → second order λ

$H' = \lambda (c \cdot \vec{x})^3$

$\langle n l m | H' | 000 \rangle = \lambda \langle n l m | (\vec{c} \cdot \vec{x})^3 | 000 \rangle$

$(\vec{c} \cdot \vec{x})^3 = (c_x \hat{x} + c_y \hat{y} + c_z \hat{z})^3$

→ Take $(c_x \hat{x} + c_y \hat{y} + c_z \hat{z})^3$

$a_+ = \frac{-1}{\sqrt{2}} (a_x - i a_y)$

$a_- = \frac{1}{2} (a_x + i a_y)$

$$\left[\frac{c_x}{\sqrt{2}} (a_- - a_+ + a_-^+ - a_+^+) - \frac{c_y i}{\sqrt{2}} (a_- + a_+ - a_-^+ - a_+^+) + \dots \right.$$

$$\left. \dots c_z (a_- - a_+ + a_-^+ - a_+^+) \right]^3$$

$\Rightarrow x = \frac{1}{\sqrt{2}} [a_- - a_+ + a_-^+ - a_+^+]$

$y = \frac{-i}{\sqrt{2}} [a_- + a_+ - a_-^+ - a_+^+]$

$z = (a_- - a_+ + a_-^+ - a_+^+)$

X: $(a_- - a_+ + a_-^+ - a_+^+)^2 = a_-^2 + a_+^2 + a_-^{+2} + a_+^{+2} + a_- a_-^+ + a_-^+ a_- + a_+ a_+^+ + a_+^+ a_+$

$= a_-^2 + a_+^2 + a_-^{+2} + a_+^{+2} + 2a_- a_-^+ + 2 a_+ a_+^+ + 2$

$[a_+, a_+^+] = [a_-, a_-^+] = 1$

$[a_+, a_-] = [a_+, a_-^+] = [a_+^+, a_-] = [a_+^+, a_-^+] = 0$

Now cubed: $a_-^3 + a_+^3 + a_-^{+3} + a_+^3 + 2 a_- a_-^+ a_- + 2 a_- a_-^{+2} + \dots$

$\dots + 2 a_+ a_+^+ a_+ + 2 a_+ a_+^{+2} + 2$

$(a+b+c)^3 = a^3 + 3a^2 b + 3a^2 c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2 c + 3bc^2 + c^3$

$\Rightarrow a_-^{+3} + a_+^{+3} + 2 a_- a_-^{+2} - 2 a_+ a_+^2 + 2 (a_+^+ - a_-^+)$

Y: $(a_- + a_+ - a_-^+ - a_+^+)^3$

$\Rightarrow -a_-^{+3} - a_+^3 + 2 a_- a_-^{+2} + 2 a_+ a_+^{+2} + 2 (-a_+^+ - a_-^+)$

Z: $(a_- - a_+ + a_-^+ - a_+^+)^3$

$\Rightarrow a_-^{+3} - a_+^{+3} + 2 a_- a_-^{+2} - 2 a_+ a_+^{+2} + 2 (-a_+^+ + a_-^+)$

$x^3 + y^3 + z^3 = a_-^{+3} + a_+^3 + \underset{\downarrow\, 2}{6}\, a_- a_-^{+2} - 2 a_+ a_+^{+2} + 2 (-a_+^+ + a_-^+)$

Final states

$|3,0,0\rangle$, $|030\rangle$ $|003\rangle$

$|210\rangle$ $|201\rangle$ $|120\rangle$ $|102\rangle$ $|021\rangle$

$|111\rangle$

$\Bigg\}\ \delta E = \frac{9}{2} \hbar\omega - \frac{3}{2} \hbar\omega = 3 \hbar\omega$

$|100\rangle$ $|010\rangle$ $|001\rangle$ → $\Delta E = \frac{5}{2} \hbar\omega - \frac{3}{2} \hbar\omega = \hbar\omega$

→ Now to get $|3,0,0\rangle$, $|030\rangle$ or $|003\rangle$ we get $\sqrt{3 \cdot 2} = \sqrt{6}$ as constant

For $|210\rangle$ & permutations we get $\sqrt{2}$ and for $|111\rangle$ we get 1

→ For $|100\rangle$ & permutations $\sqrt{2}$

$$\frac{6}{3\hbar\omega} (c_x^6 + c_y^6 + c_z^6) + \frac{2}{3\hbar\omega} c_x^4 c_y^2 + c_x^2 c_z^2 + \dots$$