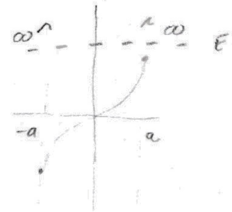
QM-4 January 2015



$$V(x) = \begin{cases} \infty & |x| > a \\ \beta x^3 & |x| < a \end{cases}$$

$E > \beta a^3$

WKB wave function

$$\psi(x) = \frac{C}{\sqrt{p(x)}} \sin\left(\frac{1}{\hbar}\int_{-a}^{x} p(y)\,dy\right)$$

a. Derive Bohr-Sommerfeld quantization rule

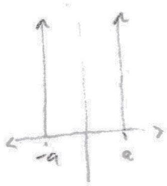
$\psi(x=a)=0$ → boundary condition for this potential

$$\psi(a) = \frac{C}{\sqrt{p(x)}} \sin\left(\frac{1}{\hbar}\int_{-a}^{a} p(y)\,dy\right) = 0$$

$$\Rightarrow \sin\left(\frac{1}{\hbar}\int_{-a}^{a} p(y)\,dy\right) = 0 \qquad \Rightarrow \frac{1}{\hbar}\int_{-a}^{a} p(y)\,dy = n\pi$$

$$\rightarrow \int_{-a}^{a} p(y)\,dy = \hbar n \pi$$

b. Find WKB energy levels at $\beta = 0$



→ infinite square well

$p(x) = \sqrt{2m(E-V)}$ $\qquad V=0$

$p(x) = \sqrt{2mE}$

$$\int_{-a}^{a} p(y)\,dy \Rightarrow \int_{-a}^{a} p(x)\,dx = \hbar n\pi \Rightarrow p(x)a - (-p(x)a) = 2a\,p(x)$$

$$p(x) = \frac{\hbar n \pi}{2a}$$

$$E = \frac{p^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{8a^2 m}$$

c. Small β, find WKB levels up to β^2

$$p(x) = \sqrt{2m(E-V)} = \left[2mE - 2m\beta x^3\right]^{\frac{1}{2}} = \sqrt{2mE}\,\underbrace{\left[1 - \frac{\beta x^3}{E}\right]^{\frac{1}{2}}}_{1 - \frac{1}{2}\frac{\beta x^3}{E} + \frac{1}{8}\left(\frac{\beta^2 x^6}{E^2}\right)}$$

$$p(x) \sim \sqrt{2mE}\left[1 - \frac{1}{2}\frac{\beta x^3}{E} + \frac{1}{8}\left(\frac{\beta^2 x^6}{E^2}\right)\right]$$

$$\int_{-a}^{a} p(y)\,dy = \hbar n \pi$$

$$\Rightarrow \sqrt{2mE}\int_{-a}^{a}\left[1 - \frac{1}{2}\frac{\beta x^3}{E} + \frac{1}{8}\left(\frac{\beta^2 x^6}{E^2}\right)\right]dx = \sqrt{2mE}\left[\underbrace{\int_{-a}^{a} dx}_{2a} + \frac{-\beta}{2E}\underbrace{\int_{-a}^{a} x^3\,dx}_{0} + \frac{\beta^2}{8E^2}\underbrace{\int_{-a}^{a} x^6\,dx}_{\frac{x^7}{7} \to \frac{a^7}{7} - \frac{-a^7}{7} = \frac{2a^7}{7}}\right]$$

$$\hbar n \pi = \sqrt{2mE}\left(2a + \frac{\beta^2}{8E^2}\frac{2a^7}{7}\right) = \sqrt{2mE}\left(2a + \frac{\beta^2 a^7}{28E^2}\right)$$

$$\sqrt{2mE} = \frac{\hbar n \pi}{2a + \frac{\beta^2 a^7}{28E^2}} = \frac{28E^2 \hbar n \pi}{56aE^2 + \beta^2 a^7}$$

$$2mE = E^4\left(\frac{28\hbar n\pi}{56aE^2 + \beta^2 a^7}\right)^2 \Rightarrow E^{-3} = \frac{1}{2m}\left(\frac{28\hbar n\pi}{56aE^2 + \beta^2 a^7}\right)^2$$

$$E^3 = 2m\left(\frac{56aE^2 + \beta^2 a^7}{28\hbar n\pi}\right)^2 = 2m\left(\frac{2aE^2}{\hbar n \pi} + \frac{\beta^2 a^7}{28\hbar n\pi}\right)^2$$

Check $\beta \to 0$ we get E_0

$$E^3 = 2m\left(\frac{2aE^2}{\hbar n\pi}\right)^2 = \frac{8ma^2E^4}{(\hbar n \pi)^2} \Rightarrow E = \frac{\hbar^2 n^2 \pi^2}{8ma^2}$$

$\to$ we recover b for $\beta \to 0$:)