QM-2 January 2016

Two - level quantum system

$|\Phi\rangle = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$ $\quad \hat{H} = \lambda \hat{I} - u_1 \sigma_x - u_2 \sigma_y \quad \rightarrow \hat{H} = \begin{bmatrix} \lambda & -u_1 + iu_2 \\ -u_1 - iu_2 & \lambda \end{bmatrix}$

a. What are energy eigenvalues?

$$\left| \begin{bmatrix} \lambda - \varepsilon & -u_1 + iu_2 \\ -u_1 - iu_2 & \lambda - \varepsilon \end{bmatrix} \right| = (\lambda - \varepsilon)^2 - (-u_1 + iu_2)(-u_1 - iu_2) = 0$$

$$= (\lambda - \varepsilon)^2 - (u_1^2 + u_2^2) = \varepsilon^2 - 2\lambda\varepsilon + (\lambda^2 - u_1^2 - u_2^2) = 0$$

$$\varepsilon = \frac{2\lambda \pm \sqrt{4\lambda^2 - 4\lambda^2 + 4u_1^2 + 4u_2^2}}{2} \qquad \varepsilon = \frac{2\lambda}{2} \pm \frac{1}{2}\sqrt{4(u_1^2 + u_2^2)}$$

$$\varepsilon^1 = \lambda + \sqrt{u_1^2 + u_2^2} \qquad \varepsilon^2 = \lambda - \sqrt{u_1^2 + u_2^2}$$

b. What are the eigenvectors? $\quad u_1 = u\cos\alpha \quad u_2 = u\sin\alpha$

- ε^1 :

$$\begin{bmatrix} -\sqrt{u_1^2 + u_2^2} & -u_1 + iu_2 \\ -u_1 - iu_2 & -\sqrt{u_1^2 + u_2^2} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$u_1^2 + u_2^2 = u^2$

$-u_1 + iu_2 = u(-\cos\alpha + i\sin\alpha)$
$= -u e^{-i\alpha}$

$-u_1 - iu_2 = -u e^{i\alpha}$

$$\begin{bmatrix} -u & -u e^{-i\alpha} \\ -u e^{i\alpha} & -u \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Phi_1 + \Phi_2 e^{-i\alpha} = 0 \qquad \Phi_1 = -e^{-i\alpha}\Phi_2$

$|\Phi_1|^2 + |\Phi_2|^2 = 1 \qquad 2|\Phi_2|^2 = 1$
$\Phi_2 = \frac{1}{\sqrt{2}}$

$$|E^1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} e^{i\alpha} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -e^{i\alpha} \end{bmatrix}$$

- ε^2

$$\begin{bmatrix} \sqrt{u_1^2 + u_2^2} & -u_1 + iu_2 \\ -u_1 - iu_2 & \sqrt{u_1^2 + u_2^2} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u & -u e^{-i\alpha} \\ -u e^{i\alpha} & u \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$$

$\Phi_1 - \Phi_2 e^{-i\alpha} = 0$
$\Phi_1 = \Phi_2 e^{-i\alpha}$

$$|E^2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ e^{i\alpha} \end{bmatrix}$$

c. At $t=0$ we have $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ find $|\phi(t)\rangle$ in terms of E', E^2 & $|E'\rangle, |E^2\rangle$

$|\phi(t)\rangle = \mathcal{U}(t)\, |\phi(0)\rangle$ $\qquad \mathcal{U}(t) = e^{-i\hat{H}t/\hbar}$

$|E'\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\alpha} \\ -1 \end{bmatrix}$

$|\phi(0)\rangle = c\left(|E'\rangle - |E^2\rangle\right) = \frac{c}{\sqrt{2}} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \frac{-2c}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$|E^2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\alpha} \\ 1 \end{bmatrix}$

$\left(\frac{-2}{\sqrt{2}}\right)^2 = \frac{1}{c^2} \Rightarrow \frac{4}{2} = \frac{1}{c^2} \Rightarrow \frac{1}{2} = c^2$

$|\phi(0)\rangle = \frac{1}{\sqrt{2}} \left[|E'\rangle - |E^2\rangle\right]$

$e^{-i\hat{H}t/\hbar}\, |\phi(0)\rangle = \frac{1}{\sqrt{2}} \left[e^{-iE't/\hbar} |E'\rangle - e^{-iE^2t/\hbar} |E^2\rangle\right]$

$|\phi(t)\rangle = \frac{1}{\sqrt{2}} \left[e^{-iE't/\hbar} |E'\rangle - e^{-iE^2t/\hbar} |E^2\rangle\right]$ it is already normalized

d. Probability of finding it in $|\tilde{\phi}\rangle$ at time $t>0$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c\left(|E'\rangle + |E^2\rangle\right) = \frac{c}{\sqrt{2}} \begin{bmatrix} e^{-i\alpha} + e^{-i\alpha} \\ -1+1 \end{bmatrix} = \frac{2e^{-i\alpha}}{\sqrt{2}}\, c \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad c = \frac{\sqrt{2}}{2e^{-i\alpha}} = \frac{1}{\sqrt{2}} e^{i\alpha}$

$|\tilde{\phi}\rangle = \frac{e^{i\alpha}}{\sqrt{2}} \left(|E'\rangle + |E^2\rangle\right)$

$\langle\tilde{\phi}|\phi(t)\rangle = \frac{e^{-i\alpha}}{2} \left(\langle E'| + \langle E^2|\right)\left(e^{-iE't/\hbar}|E'\rangle - e^{-iE^2t/\hbar}|E^2\rangle\right) = \frac{e^{-i\alpha}}{2}\left(e^{-iE't/\hbar} - e^{-iE^2t/\hbar}\right)$

$E' = \lambda + \mu$
$E^2 = \lambda - \mu$

$= \frac{e^{-i\alpha}}{2} e^{-i\lambda t/\hbar} \underbrace{\left(e^{-i\mu t/\hbar} - e^{i\mu t/\hbar}\right)}_{2\sin\left(\frac{\mu}{\hbar}t\right)} = e^{-i\left(\alpha + \frac{\lambda t}{\hbar}\right)} \sin\left(\frac{\mu}{\hbar}t\right)$

$|\langle\hat{\phi}|\phi(t)\rangle|^2 = \sin^2\left(\frac{\mu}{\hbar}t\right)$