SM-2 January 2016

Colloidal particle in optical trap.

$$0 = -\lambda \dot{x} - kx + f(t)$$

$\lambda \to$ drag coefficient
$k \to$ optical trap stiffness
$f(t) \to$ random thermal force

$$\langle f(t) \rangle = 0$$
$$\langle f(t) f(t') \rangle = \frac{2\lambda}{\beta} \delta(t-t')$$

a. Verify following is a solution for $x(0)=0$

$$x(t) = \frac{1}{\lambda} e^{-\frac{k}{\lambda}t} \int_0^t e^{\frac{k}{\lambda}t'} f(t')\,dt'$$

$$\dot{x}(t) = \frac{1}{\lambda}\left[\left(\frac{-k}{\lambda}\right) e^{-\frac{k}{\lambda}t} \int_0^t e^{\frac{k}{\lambda}t'} f(t')\,dt' + \underbrace{e^{-\frac{k}{\lambda}t} e^{\frac{k}{\lambda}t} f(t)}_{\text{derivative of an integral}}\right]$$

$$= \frac{-k}{\lambda^2} e^{-\frac{k}{\lambda}t} \int_0^t e^{\frac{k}{\lambda}t'} f(t')\,dt' + \frac{f(t)}{\lambda}$$

$$0 = -\lambda\dot{x} - kx + f(t) = \frac{k}{\lambda} e^{-\frac{k}{\lambda}t} \int_0^t e^{\frac{k}{\lambda}t'} f(t')\,dt' - f(t) - \frac{k}{\lambda} e^{-\frac{k}{\lambda}t} \int_0^t e^{\frac{k}{\lambda}t'} f(t')\,dt' + f(t)$$

$$= -f(0) + f(0) = 0 \quad \checkmark$$

b. Solve for $\langle x(t)^2 \rangle$ where $x(0)=0$ at $t=0$.

$$\langle x(t)^2 \rangle = \left\langle \frac{1}{\lambda^2} e^{-\frac{2k}{\lambda}t} \int_0^t e^{\frac{k}{\lambda}(t'+t'')} f(t') f(t'')\,dt'\,dt'' \right\rangle = \frac{1}{\lambda^2} e^{-\frac{2k}{\lambda}t} \int_0^t e^{\frac{k}{\lambda}(t'+t'')} \underbrace{\langle f(t') f(t'') \rangle}_{\frac{2\lambda}{\beta}\delta(t'-t'')} dt'\,dt''$$

$$= \frac{2}{\beta\lambda} e^{-\frac{2k}{\lambda}t} \int_0^t e^{\frac{k}{\lambda}(t'+t'')} \delta(t'-t'')\,dt'\,dt''$$

$$= \frac{2}{\beta\lambda} e^{-\frac{2k}{\lambda}t} \int_0^t e^{\frac{2k}{\lambda}t'}\,dt' = \frac{2}{\beta\lambda} e^{-\frac{2k}{\lambda}t} \left(e^{\frac{2k}{\lambda}t} - 1\right)$$

c. Compute $\frac{1}{T}\langle|\tilde{x}(\omega)|^2\rangle$

$$\tilde{x}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-T/2}^{T/2} e^{-i\omega t} x(t)\, dt$$

$$|\tilde{x}(\omega)|^2 = \frac{1}{2\pi}\int_{-T/2}^{T/2} e^{i\omega t} x^*(t)\, e^{i\omega t} x(t)\, dt = \frac{1}{2\pi}\int_{-T/2}^{T/2}\left[e^{i\omega t}\frac{1}{\lambda}e^{-\frac{k}{\lambda}t}\int_0^t e^{\frac{k}{\lambda}t'} f(t')\,dt'\right]\left[e^{-i\omega t}\frac{1}{\lambda}e^{-\frac{k}{\lambda}t}\int_0^t e^{\frac{k}{\lambda}t''} f(t'')\,dt''\right]dt$$

$$= \frac{1}{2\pi\lambda^2}\int_{-T/2}^{T/2}\left[e^{(i\omega-\frac{k}{\lambda})t}\int_0^t e^{\frac{k}{\lambda}t'} f(t')\,dt'\right]\left[e^{-(i\omega+\frac{k}{\lambda})t}\int_0^t e^{\frac{k}{\lambda}t''} f(t'')\,dt'\right]$$

$$= \frac{1}{2\pi\lambda^2}\int_{-T/2}^{T/2} e^{-\frac{2k}{\lambda}t}\int_0^t e^{\frac{k}{\lambda}(t'+t'')} f(t')f(t)\,dt'\,dt''$$

$$\langle|\tilde{x}(\omega)|^2\rangle = \frac{1}{2\pi\lambda^2}\int_{-T/2}^{T/2} e^{-\frac{2k}{\lambda}t}\int_0^t e^{\frac{k}{\lambda}(t'+t'')}\underbrace{\langle f(t')f(t'')\rangle}_{\frac{2\lambda}{\beta}\delta(t'-t'')}dt'\,dt''$$

$$= \frac{1}{\pi\beta\lambda}\int_{-T/2}^{T/2} e^{-2\frac{k}{\lambda}t}\underbrace{\int_0^t e^{\frac{2k}{\lambda}t'}\,dt'}_{\frac{\lambda}{2k}\left(e^{\frac{2k}{\lambda}t}-1\right)} = \frac{1}{2\pi\beta k}\int_{-T/2}^{T/2} e^{-2\frac{k}{\lambda}t}\left(e^{\frac{2k}{\lambda}t}-1\right)dt$$

$$= \frac{1}{2\pi\beta k}\int_{-T/2}^{T/2} 1 - e^{-\frac{2k}{\lambda}t}\,dt = \frac{-1}{2\pi\beta k}\left[\left(\frac{-\lambda}{2k}\right)\left(e^{-\frac{2k}{\lambda}\frac{T}{2}} - e^{\frac{2k}{\lambda}\frac{T}{2}}\right) + \left(T/2 + \frac{T}{2}\right)\right]$$

$$= \frac{\lambda}{4\pi\beta k^2}\left(e^{-\frac{kT}{\lambda}} - e^{\frac{kT}{\lambda}}\right) - \frac{T}{2\pi\beta k} =$$

Spectrum: $\frac{1}{T}\langle|\tilde{x}(\omega)|^2\rangle = \frac{\lambda}{4\pi\beta T k^2}\left(e^{-\frac{kT}{\lambda}} - e^{\frac{kT}{\lambda}}\right) - \frac{1}{2\pi\beta k}$