SM-2 January 2017

Black hole mass M — surface area A

internal energy $E = Mc^2$ — radius $R = \frac{2GM}{c^2}$

entropy $S = \frac{k_B c^3}{4G\hbar} A$

a. Derive expression for temperature in terms of mass

$$\frac{\partial S}{\partial E} = \frac{1}{T}$$

surface area of sphere: $4\pi r^2$

$$A = 4\pi R^2 = 4\pi\left(\frac{2GM}{c^2}\right)^2 = 16\pi\frac{G^2M^2}{c^4}$$

$$\frac{\partial}{\partial E}\left(\frac{k_B c^3}{4G\hbar}A\right) = \frac{\partial}{\partial(Mc^2)}\left[\frac{k_B c^3}{4G\hbar}A\right]$$

$$= \frac{\partial}{\partial(Mc^2)}\left[\frac{k_B c^3}{4G\hbar}16\pi\frac{G^2M^2}{c^4}\right] = \frac{\partial}{\partial(Mc^2)}\left[\frac{4\pi k_B G M^2}{\hbar c}\right] = \frac{4\pi k_B G}{\hbar}\frac{\partial}{\partial(Mc^2)}\left[\frac{(Mc^2)^2}{c(c^2)^2}\right]$$

$$= \frac{4\pi k_B G}{\hbar}\left[\frac{2(Mc^2)}{c^5}\right] = \frac{8\pi k_B G M}{\hbar c^3} = \frac{1}{T}$$

$$T = \frac{\hbar c^3}{8\pi k_B G M}$$

b. Radiated power per unit surface area j

$$j = \sigma T^4 \qquad \sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$$

Find rate of energy loss → total power

$$P = \frac{E}{\Delta t}$$

$$j = \frac{\pi^2 k_B^4}{60\hbar^3c^2}\cdot\frac{\hbar^4 c^{12}}{(8)^4 k_B^4 G^4 M^4 \pi^4} = \frac{\hbar c^{10}}{60(8)^4 G^4 M^4 \pi^2}$$

$$\int j \cdot dA = jA$$

$$jA = \frac{\hbar c^{10}}{60(8)^3 G^4 M^4 \pi^2}\cdot\frac{16\pi G^2M^2}{c^4} = \frac{\hbar c^6}{30(8)^3 G^2 M^2 \pi}$$

units: $[\hbar] = m^2\,kg/s$

$[c] = m/s$

$[G] = m^3/kg\,s^2$

$[M] = kg$

$$\frac{m^2\,kg}{s}\,\frac{m^6}{s^6}\,\frac{kg^2 s^4}{m^6}\,\frac{1}{kg^2}$$

$$\frac{m^2\,kg}{s^3} = \left[\frac{m^2\,kg}{s^2}\right]\frac{1}{s}$$

↓ energy/second

✓ units check

c. Find time it takes to evaporate:

$$P = \frac{-dE}{dt}$$

$$dt = \frac{-dE}{P} \qquad dE = c^2\, dM$$

$$\int_0^t dt = \int_M^0 -\frac{c^2}{P}\, dM$$

$$t = -\int_M^0 c^2 \frac{30(8)^3 G^2 M^2 \pi}{\hbar c^6}\, dM$$

$$= \frac{-30(8)^3 G^2 \pi}{\hbar c^4} \int_M^0 M^2\, dM$$

$$= \frac{10(8)^3 G^2 \pi}{\hbar c^4} M^3$$

$[M] = kg$ $\qquad g$

$[G] = m^3/kg\,s$ $\qquad cm^3/g\,s^2$

$[\hbar] = m^2\,kg/s$

$[c] = m/s$

$$\to \frac{kg^3 \left[m^3/kg\,s^2\right]^2}{\frac{m^2\,kg}{s} \cdot \frac{m^4}{s^4}}$$

$$\frac{s^5\, kg^3\, m^6}{kg^3\, s^4\, m^6} = s$$

✓ units work.