QM-3 September 2014

2D oscillator

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(1 + \delta \cdot x \cdot y)(x^2 + y^2)$$

$\hbar = 1$ & $m = 1$ & $\omega^2 = 1$

a. Find wavefunctions & energies for 3 lowest states when $\delta = 0$

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2)$$

$H_x = \frac{1}{2} p_x^2 + \frac{1}{2} x^2$ & $H_y = \frac{1}{2} p_y^2 + \frac{1}{2} y^2$

$E_x = \frac{1}{2} + n_x$ $E_y = (\frac{1}{2} + n_y)\hbar\omega$

$$E = (1 + n_x + n_y)\hbar\omega$$

$$\left.\begin{array}{l} |00\rangle \Rightarrow E_{00} = \hbar\omega = 1 \\ |10\rangle \\ |01\rangle \end{array} \begin{array}{l} \\ \Rightarrow E_{10} = 2\hbar\omega = 2 \\ E_{01} = 2\hbar\omega = 2 \end{array}\right\}$$ 3 lowest energy states

$\psi(x,y) = \psi(x)\psi(y)$

$\psi_0(x) = A e^{-\omega x^2/2} \rightarrow |\psi|^2 = A^2 \int_{-\infty}^{\infty} e^{-\omega x^2} = A^2 \sqrt{\frac{\pi}{\omega}} = A^2\sqrt{\pi} = 1$

$A = \pi^{-1/4}$

$\psi_{00}(x,y) = \frac{1}{\sqrt{\pi}} e^{-(x^2+y^2)/2}$

$\psi_0(y) = \pi^{-1/4} e^{-y^2/2}$

Now for $\psi_{01}(x,y)$

$\psi_1(y) = Bx\, e^{-\omega x^2/2} \Rightarrow |\psi|^2 = B^2 \int_{-\infty}^{\infty} x^2 e^{-\omega x^2} dx$

$= B^2 \frac{\sqrt{\pi}}{2\omega^{3/2}} = \frac{B^2\sqrt{\pi}}{2} = 1$

$\psi_{01} = \psi_0(x)\psi_1(y) = \sqrt{\frac{2}{\pi}}\, y\, e^{-(x^2+y^2)/2}$

$B = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{(\pi)^{1/4}}$

$\psi_{10} = \psi_1(x)\psi_0(y) = \sqrt{\frac{2}{\pi}}\, x\, e^{-(x^2+y^2)/2}$

b. Find 3 lowest states for $\delta \neq 0$ using perturbation theory. (First order)

$\psi_{00} \rightarrow$ non-degenerate $\qquad H' = \frac{1}{2}\delta xy(x^2+y^2) = \frac{\delta}{2}(x^3y + xy^3)$

$$E_0^{(1)} = \langle 00|H'|00\rangle = \frac{\delta}{2}\frac{1}{\pi}\underbrace{\int_{-\infty}^{\infty} e^{-(x^2+y^2)}(x^3y+y^3x)\,dx\,dy}$$

$$\int_{-\infty}^{\infty} \underbrace{x^3 e^{-x^2}}_{\text{odd}}\,dx \int_{-\infty}^{\infty} \underbrace{y e^{-y^2}}_{\text{even}}\,dy + \int_{-\infty}^{\infty} x e^{-x^2}\,dx \int_{-\infty}^{\infty} \underbrace{y^3}_{\text{odd}}\underbrace{e^{-y^3}}_{\text{even}}\,dy = 0 \qquad (\text{odd} \cdot \text{even} = \text{even} \Rightarrow 0)$$

$= 0 \quad \rightarrow$ correction to ground state $= 0$

E'_{01} & E'_{10} $\rightarrow$ W matrix

$$W_{11} = \langle 01|H'|01\rangle = \frac{\delta}{\pi}\langle 01|x^3y + xy^3|01\rangle = \frac{\delta}{2}\left[\int x^5 e^{-x^2}dx \int y e^{-y^2}dy + \int x^3 e^{-x^2}dx \int y^3 e^{-y^3}dy\right] = 0$$

$$W_{12} = W_{21} = \langle 01|H'|10\rangle = \frac{\delta}{\pi}\left[\underbrace{\int x^4 e^{-x^2}dx}_{\frac{3\sqrt{\pi}}{4}} \underbrace{\int y^2 e^{-y^2}dy}_{\frac{\sqrt{\pi}}{2}} + \underbrace{\int x^2 e^{-x^2}dx}_{\frac{\sqrt{\pi}}{2}} \underbrace{\int y^4 e^{-y^2}dy}_{\frac{3\sqrt{\pi}}{4}}\right] = \frac{\delta}{2}\left(\frac{3\pi}{4}\right) = \frac{3\delta}{4}$$

$W_{22} = \langle 10|H'|10\rangle = 0$ (similar to W_{11})

$$W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{3\delta}{4} \qquad \text{eigenvalues: } \pm\frac{3\delta}{4}$$

eigenvectors: $\vec{V}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \rightarrow \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle\right]$

$\vec{V}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \rightarrow \frac{1}{2}\left[|01\rangle - |10\rangle\right]$

$$|00\rangle = \frac{1}{\sqrt{\pi}} e^{-(x^2+y^2)/2}$$

$$|01\rangle = |01\rangle^0 + \frac{3\delta}{4\sqrt{2}}\left[|01\rangle + |10\rangle\right] = \sqrt{\frac{2}{\pi}}\, y e^{-(x^2+y^2)/2} + \frac{3\delta}{4\sqrt{\pi}}\left[y e^{-(x^2+y^2)} + x e^{-(x^2+y^2)}\right]$$

$$|10\rangle = |10\rangle^0 - \frac{3\delta}{4\sqrt{2}}\left[|01\rangle - |10\rangle\right] = \sqrt{\frac{2}{\pi}}\, x e^{-(x^2+y^2)/2} - \frac{3\delta}{4\sqrt{\pi}}\left[y e^{-(x^2+y^2)} - x e^{-(x^2+y^2)}\right]$$