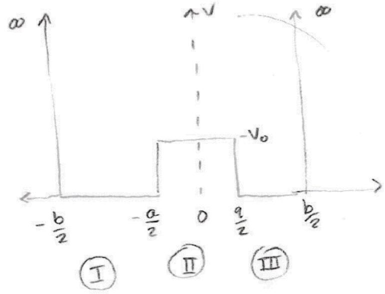
QM-2 September 2015

Assume $2mV_0a^2 \gg \hbar^2$

a. Determine energy levels for $E \ll V_0$ up to order $\frac{\hbar^2}{2mV_0a^2}$



3 Regions:

(I) $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi = E\Psi \rightarrow \Psi_I = A\sin(Kx) + B\cos(Kx)$

(II) $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi + V_0\Psi = E\Psi \rightarrow \Psi_{II} = Ce^{ikx} + De^{-ikx}$

(III) $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi = E\Psi \rightarrow \Psi_{III} = E\sin(Kx) + F\cos(Kx)$

$K = \frac{\sqrt{2mE}}{\hbar}$

$k = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

Boundary conditions

1). At $-\frac{b}{2}$: $\Psi_I(-\frac{b}{2}) = 0$

3) At $\frac{b}{2}$: $\Psi_{III}(\frac{b}{2}) = 0$

2). At $-\frac{a}{2}$: $\Psi_I(\frac{-a}{2}) = \Psi_{II}(\frac{-a}{2})$

$\frac{d}{dx}\Psi_I(\frac{-a}{2}) = \frac{d}{dx}\Psi_{II}(\frac{-a}{2})$

4) At $\frac{a}{2}$: $\Psi_{II}(\frac{a}{2}) = \Psi_{III}(\frac{a}{2})$

$\frac{d}{dx}\Psi_{II}(\frac{a}{2}) = \frac{d}{dx}\Psi_{III}(\frac{a}{2})$

By symmetry of potential, we know solutions to wavefunctions should be symmetric or anti-symetric. Ground state will be symmetric and have no nodes.

Symmetric wave-functions:

$-A = E \qquad B = F$ and $C = D$

$\Psi_{II} = C\cosh(kx) \qquad \Psi_I = \Psi_{III} = A\sin(Kx) + B\cos(Kx)$

BC:

(4)

$-A\sin(\frac{aK}{2}) + B\cos(\frac{aK}{2}) = C\cosh(k^{a}/_{2}) \qquad -AK\cos(\frac{aK}{2}) - BK\sin(\frac{aK}{2}) = Ck\sinh(k\frac{a}{2})$

(3) $-A\sin(\frac{bK}{2}) + B\cos(\frac{bK}{2}) = 0$

$B = A\tan(\frac{bK}{2})$

$\Rightarrow -A\sin(\frac{aK}{2}) + A\tan(\frac{bK}{2})\cos(\frac{aK}{2}) = C\cosh(k^{a}/_{2})$

$A(\tan(\frac{bK}{2})\cos(\frac{aK}{2}) - \sin(\frac{aK}{2})) = C\cosh(\frac{ka}{2})$

$C = \frac{A[\tan(\frac{bK}{2})\cos(\frac{aK}{2}) - \sin(\frac{aK}{2})]}{\cosh(\frac{ka}{2})}$

$\Rightarrow -AK\cos(\frac{aK}{2}) - BK\sin(\frac{aK}{2}) = \frac{A[\tan(\frac{bK}{2})\cos(\frac{aK}{2}) - \sin(\frac{aK}{2})]k\sinh(k^{a}/_{2})}{\cosh(\frac{ka}{2})}$

→ continued.

$$-Ak\cos\left(\frac{ak}{2}\right) - A\tan\left(\frac{bk}{2}\right)k\sin\left(\frac{ak}{2}\right) = A\left[\tan\left(\frac{bk}{2}\right)\cos\left(\frac{ak}{2}\right) - \sin\left(\frac{ak}{2}\right)\right] \kappa \tanh\left(\frac{\kappa a}{2}\right)$$

$$\Rightarrow -k\left[\cos\left(\frac{ak}{2}\right) + \tan\left(\frac{bk}{2}\right)\sin\left(\frac{ak}{2}\right)\right] = \kappa\left[\tan\left(\frac{bk}{2}\right)\cos\left(\frac{ak}{2}\right) - \sin\left(\frac{ak}{2}\right)\right] \tanh\left(\frac{\kappa a}{2}\right)$$

$$-\frac{k}{\kappa} = \frac{\tanh\left(\frac{\kappa a}{2}\right)\left[\tan\left(\frac{bk}{2}\right)\cos\left(\frac{ak}{2}\right) - \sin\left(\frac{ak}{2}\right)\right]}{\left[\cos\left(\frac{ak}{2}\right) + \tan\left(\frac{bk}{2}\right)\sin\left(\frac{ak}{2}\right)\right]}$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$\kappa = \frac{\sqrt{2m(E-V_0)}}{\hbar}$$

$$-\sqrt{\frac{2mE}{2m(E-V_0)}} = -\sqrt{\frac{E}{(E-V_0)}} = \frac{\tanh\left(\frac{\kappa a}{2}\right)\left[\tan\left(\frac{bk}{2}\right)\cos\left(\frac{ak}{2}\right) - \sin\left(\frac{ak}{2}\right)\right]}{\left[\cos\left(\frac{ak}{2}\right) + \tan\left(\frac{bk}{2}\right)\sin\left(\frac{ak}{2}\right)\right]}$$

$$\frac{E}{E-V_0} = \tanh^2\left(\frac{\kappa a}{2}\right)\frac{\left[\tan\left(\frac{bk}{2}\right)\cos\left(\frac{ak}{2}\right) - \sin\left(\frac{ak}{2}\right)\right]^2}{\left[\cos\left(\frac{ak}{2}\right) + \tan\left(\frac{bk}{2}\right)\sin\left(\frac{ak}{2}\right)\right]^2}$$

$$\frac{\kappa a}{2} = \tilde{a}$$

$$\frac{bk}{2} = b$$

$$\frac{ak}{2} = a$$

$$\frac{E}{E-V_0} = \tanh^2(\tilde{a})\frac{\left[\tan^2(b)\cos^2(a) - 2\tan(b)\cos(a)\sin(a) + \sin^2(a)\right]}{\left[\cos^2(a) + 2\cos(a)\tan(b)\sin(a) + \tan^2(b)\sin^2(a)\right]}$$

$\rightarrow$

$$E\cos^2(a) + 2E\cos(a)\tan(b)\sin(a) + E\tan^2 b\sin^2 a = (E-V_0)\tanh^2(\tilde{a})\left[\tan^2 b\cos^2 a - 2\tan(b)\cos a\sin a + \sin^2 a\right]$$

$$E\left[\cos^2 a + 2\cos a\tan b\sin a + \tan^2 b\sin^2 a - (\tanh^2 a)(\tan^2 b\cos^2 a - 2\tan b\cos a\sin a + \sin^2 a)\right] = -V_0\tanh^2(\tilde{a})\left[\quad\right]$$

$$E = \frac{-V_0\tanh^2(\tilde{a})\left[\tan^2 b\cos^2 a - \frac{2}{2}\tan b\sin(2a) + \sin^2(a)\right]}{\cos^2 a + \tan b\sin(2a) + \tan^2 b\sin^2 a - (\tanh^2 a)(\tan^2 b\cos^2 a - \tan b\sin(2a) + \sin^2(a))}$$