QM-4 September 2015

Coherent states $[\hat{a}, \hat{a}^+] = 1$

a. Prove coherent state is eigenstate of $\hat{a}$.
Find its eigenvalue

Rewriting $|z\rangle$:

$$|z\rangle = e^{z\hat{a}^+}|0\rangle = \sum_{n=0}^{\infty} \frac{z^n}{n!}(a^+)^n|0\rangle$$

$$|z\rangle \equiv e^{z\hat{a}^+}|0\rangle$$

$$\hat{a}|z\rangle = \hat{a}\left(e^{z\hat{a}^+}|0\rangle\right) = \hat{a}\sum_{n=0}^{\infty}\frac{z^n}{n!}(\hat{a}^+)^n|0\rangle$$

$$= \sum_{n=1}^{\infty}\frac{z^n}{n!}(\hat{a}\hat{a}^+)(\hat{a}^+)^{n-1}|0\rangle$$

$$\hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$$

$$\hat{a}\hat{a}^+ = 1 + \hat{a}^+\hat{a}$$

$$= \sum_{n=1}^{\infty}\frac{z^n}{n!}(1+\hat{a}^+a)(\hat{a}^+)^{n-1}|0\rangle = \sum_{n=1}^{\infty}\left[\frac{z^n}{n!}(\hat{a}^+)^{n-1} + \frac{z^n}{n!}(\hat{a}^+a)(\hat{a}^+)^{n-1}\right]|0\rangle$$

$$= \sum_{n=1}^{\infty} z\left(\frac{z^{n-1}}{n!}(\hat{a}^+)^{n-1}\right)|0\rangle + \sum_{n=2}^{\infty}\frac{z^n}{n!}\hat{a}^+(1+\hat{a}^+\hat{a})(\hat{a}^+)^{n-2}|0\rangle$$

$$\vdots$$

$$= \sum_{n=1}^{\infty} nz\left(\frac{z-1}{n!}\right)^{n-1}(\hat{a}^+)^{n-1}|0\rangle + \underbrace{\sum_{n=0}^{\infty}\frac{z^n}{n!}(\hat{a}^+)^n\hat{a}|0\rangle}_{=0}$$

$$= \sum_{n=1}^{\infty} z\frac{(z-1)^{n-1}}{(n-1)!}(\hat{a}^+)^{n-1}|0\rangle = z\underbrace{\sum_{n=1}^{\infty}\frac{(z-1)^{n-1}}{(n-1)!}(\hat{a}^+)^{n-1}|0\rangle}_{|z\rangle}$$

$\Rightarrow \hat{a}|z\rangle = z|z\rangle$ $\rightarrow$ eigenvalue z ✓.

b. Does $\hat{a}^+$ have an eigenstate?

No, it raises $|0\rangle$ to $|1\rangle$ inside the definition so we don't have an eigenstate anymore (we would have $e^{z\hat{a}^+}|1\rangle$ instead).

c. Evaluate $\langle z_1 | z_2 \rangle$. Normalize $|z\rangle$. Evaluate $\langle \hat{N} \rangle$

$\langle z_1 | = \langle 0 | \exp[z^* \hat{a}]$ $\qquad |z_2\rangle = \exp[z \hat{a}^+] |0\rangle$

$$\langle z_1 | z_2 \rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \langle 0 | \frac{z^{*n_1}}{n_1!} (\hat{a})^{n_1} \frac{z^{n_2}}{n_2!} (\hat{a}^+)^{n_2} |0\rangle$$

For $n_1 \neq n_2 \Rightarrow \langle 0 | \hat{a}^{n_1} \hat{a}^{n_2} | 0 \rangle = 0$

$$= \sum_{n_1} \sum_{n_2} \frac{z^{*n_1}}{n_1!} \frac{z^{n_2}}{n_2!} \langle 0 | \hat{a}^{n_1} \hat{a}^{+n_2} | 0 \rangle$$

Evaluate $\sum_n \langle 0 | (\hat{a} \hat{a}^+)^n | 0 \rangle$

$\langle 0 | \hat{a} \hat{a}^+ | 0 \rangle = 1 \langle 0 | 0 \rangle = 1$

$\langle 0 | \hat{a}^2 \hat{a}^{+2} | 0 \rangle = \langle 0 | a^2 a^+ | 1 \rangle = \sqrt{2} \langle 0 | a^2 | 2 \rangle$
$= 2 \langle 0 | a | 1 \rangle = 2 \langle 0 | 0 \rangle = 2$

$\langle 0 | \hat{a}^3 \hat{a}^{+3} | 0 \rangle = \sqrt{6} \langle 0 | a^3 | 3 \rangle = 6$

$\vdots$

$\langle 0 | \hat{a}^n \hat{a}^{+n} | 0 \rangle = n!$

For $n_1 = n_2$ ($z_1 = z_2$)

$$\langle z_1 | z_1 \rangle = \sum_{n_1=0}^{\infty} \frac{(z^* z)^n}{(n!)^2} \langle 0 | (\hat{a} \hat{a}^+)^n | 0 \rangle$$

$$= \sum_{n=0}^{\infty} \frac{(z^* z)^n}{(n!)^2} n! = \sum_{n=0}^{\infty} \frac{(z^* z)^n}{n!}$$

$\langle z_1 | z_1 \rangle = e^{z^* z}$ $\qquad \Rightarrow \langle z | z \rangle = e^{|z|^2}$

$\langle z | \hat{a}^+ \hat{a} | z \rangle = z \langle z | \hat{a}^+ | z \rangle$

$= z^* z \langle z | z \rangle$ $\qquad \langle z | \hat{a}^+ = z^* \langle z |$

$$\langle N \rangle = \frac{\langle z | \hat{a}^+ \hat{a} | z \rangle}{\langle z | z \rangle} = z^* z \frac{\langle z | z \rangle}{\langle z | z \rangle} = |z|^2$$

d. Find time evolution of coherent state.

$\hat{H} = \hbar \omega \hat{a}^+ \hat{a}$.

$$\mathcal{U}(t) |z\rangle = e^{-i\hat{H}t/\hbar} |z\rangle = e^{-i\omega t (\hat{a}^+ a)} |z\rangle = e^{-i\omega t (\hat{a}^+ \hat{a})} \sum_{n=0}^{\infty} \frac{z^n}{n!} (\hat{a}^+)^n |0\rangle = \sum_{n=0}^{\infty} \frac{(-i\omega t)^n}{n!} \frac{z^n}{n!} \underbrace{(\hat{a}^+ \hat{a})^n (\hat{a}^+)^n |0\rangle}$$

$(\hat{a}^+ \hat{a})^n |0\rangle = 1^n |0\rangle$

$\rightarrow \hat{a}^{+n} (\hat{a}^+ \hat{a})^n |0\rangle = (\hat{a}^+)^n |0\rangle$

$\hat{a}^+ \hat{a} \hat{a}^+ - \hat{a}^+ \hat{a} \hat{a}^+ = 0$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{(-i\omega t)^n}{n!}}_{e^{-i\omega t}} \underbrace{\frac{z^n}{n!} \hat{a}^{+n} |0\rangle}_{\epsilon} = \sum_{n=0}^{\infty} \frac{(-i\omega t)^n}{n!} z^n \hat{a}^n |0\rangle$$

$= |e^{-i\omega t} z\rangle$