QM-1 September 2017

2 level atom states $|1\rangle$ and $|2\rangle$

$\hbar\omega_{12} \equiv E_2 - E_1$ energy separation

Initially at $|1\rangle$

At time t: $\mathcal{E} = \mathcal{E}_0 (e^{i\omega t} + e^{-i\omega t})$ in x direction

$\omega \to$ oscillation frequency

$|\mathcal{E}_0| \to$ oscillation amplitude

a. Write Hamiltonian in terms of $\hat{H}_0$ and $\hat{V}$

$$\hat{H} = \hat{H}_0 + \hat{H}'$$

$$\hat{H}' = q\mathcal{E}x = q\mathcal{E}_0 (e^{i\omega t} + e^{-i\omega t}) x = \hat{V}(t) \rightarrow \hat{H} = \hat{H}^0 + \hat{V}(t)$$

b. Write the general form of the solution to $\hat{H}$

$\hat{H}|\psi\rangle = i\hbar \frac{\partial}{\partial t}|\psi\rangle$, $|\psi\rangle$ is a linear combination of $|1\rangle$ and $|2\rangle$

$$|\psi\rangle = A(t)\, e^{-i\hat{H}^0 t/\hbar}|1\rangle + B(t)\, e^{-i\hat{H}^0 t/\hbar}|2\rangle$$

$$= A(t)\, e^{-iE_1 t/\hbar}|1\rangle + B(t)\, e^{-i(\hbar\omega_{12} + E_1)t/\hbar}|2\rangle$$

$$= A(t)\, e^{-iE_1 t/\hbar}|1\rangle + B(t)\, e^{-i\omega_{21} t}\, e^{-iE_1 t/\hbar}|2\rangle$$

c. Probability of finding atom in state $|2\rangle$ at $t \geq |0\rangle$. $|\mathcal{E}_0|$ is small, $\omega = \omega_{12}$

$e^{i\omega t} \to$ can be ignored. Express in terms of $W_{12} \equiv |\mathcal{E}_0|\langle 1|\hat{x}|2\rangle = W_{21} = |\mathcal{E}_0|\langle 2|\hat{x}|1\rangle$

$\Delta\omega = \omega_{21} - \omega$

$$P_{21} = |d_{21}|^2$$

$$d_{21} = \frac{-i}{\hbar}\int_0^t \langle f|\hat{V}|i\rangle e^{i\omega_{12}t'}dt' = \frac{-i}{\hbar}\int_0^t \langle 2|(\underbrace{e^{i\omega t}}_{\text{ignore}} + e^{-i\omega t})q\mathcal{E}_0\hat{x}|1\rangle e^{i\omega_{12}t'}dt'$$

$$= \frac{-i}{\hbar}\langle 2|\hat{x}|1\rangle q\mathcal{E}_0 \int_0^t e^{i(\omega_{12}-\omega)t'}dt' = -i\frac{q\mathcal{E}_0}{\hbar}\langle 2|\hat{x}|1\rangle \frac{-i}{(\omega_{21}-\omega)} e^{i(\omega_{21}-\omega)t'}\Big|_0^t$$

$$= \frac{q\mathcal{E}_0}{\hbar}\frac{\langle 2|\hat{x}|1\rangle}{\omega_{21}-\omega}\left[e^{i(\omega_{21}-\omega)t} - 1\right]$$

$$|d_{21}|^2 = \frac{q^2}{\hbar^2}\frac{W_{12}^2}{(\omega_{21}-\omega)^2}\underbrace{\left[\cos(\Delta\omega t) + i\sin(\Delta\omega t) - 1\right]\left[\cos(\Delta\omega t) - i\sin(\Delta\omega t) - 1\right]}$$

$e^{i\Delta\omega t} = \cos(\Delta\omega t) + i\sin(\Delta\omega t)$

$$\cos^2(\Delta\omega t) + \sin^2(\Delta\omega t) + 1 + \cos(\Delta\omega t)i\sin(\Delta\omega t) - \cos(\Delta\omega t)i\sin(\Delta\omega t) - \cos(\Delta\omega t) - i\sin(\Delta\omega t)$$
$$- \cos(\Delta\omega t) + i\sin(\Delta\omega t)$$

$$= 2 - 2\cos(\Delta\omega t) = 2\underbrace{(1 - \cos(\Delta\omega t))}_{2\sin^2\left(\frac{\Delta\omega t}{2}\right)}$$

$$= \frac{4q^2 W_{12}^2}{\hbar^2(\omega_{21}-\omega)^2}\sin^2\left(\frac{\omega_{21}-\omega}{2}t\right)$$

If $\omega_{21} = \omega$:

$$d_{21} = \frac{-iq\varepsilon_0}{\hbar} \langle 2|\hat{V}|1\rangle \int_0^t dt = \frac{-iq\varepsilon_0}{\hbar} \langle 2|\hat{V}|1\rangle t$$

$$|d_{21}|^2 = \frac{q^2 W_{12}^2}{\hbar^2} t^2$$

d. Probability if $\omega_{12} \neq \omega$

I solved for this in part c

$$|d_{21}|^2 = \frac{4q^2 W_{12}^2}{\hbar^2(\omega_{21}-\omega)^2} \sin^2\left(\frac{\omega_{21}-\omega}{2} t\right)$$