SM-2 September 2017

Phonons $\rightarrow$ Bose-Einstein statistics.

$\epsilon = \hbar\omega$

$$g(\omega) = \frac{V\omega^2}{2\pi^2 c_s^3}$$ $g(\omega) \rightarrow$ density of states

$$c_s = \omega_{max}\left(\frac{6\pi^2 N}{V}\right)^{-\frac{1}{3}}$$ * Note: equation in exam is wrong, power $-\frac{1}{3}$ to get correct units.

a. Write expression for $\langle E\rangle$

$$Z_1 = \frac{1}{1 - e^{-\beta(\epsilon-\mu)}}$$ $\rightarrow$ B.E partition function $\mu = 0$

useful: $\Phi = -\frac{1}{\beta}\ln Z = -\frac{1}{\beta}\int d\epsilon\, g(\epsilon)\ln Z_\epsilon$

$$Z_T = \prod_i Z_i$$

$$g(\omega) = \frac{V\omega^2}{2\pi^2\omega_{max}^3}\left(\frac{6N\pi^2}{V}\right) = \frac{3\omega^2 N}{\omega_{max}^3}$$

$$\ln Z_T = -\sum_i \ln(1-e^{-\beta\epsilon}) = -\int_0^{\omega_{max}} g(\omega)\ln(1-e^{-\beta\epsilon})\,d\omega = \frac{-3N}{\omega_{max}^3}\int_0^{\omega_{max}}\omega^2\ln(1-e^{-\beta\hbar\omega})\,d\omega$$

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z_T$$

$$= \frac{-\partial}{\partial\beta}\left(\frac{-3N}{\omega_{max}^3}\right)\int_0^{\omega_{max}}\omega^2\ln(1-e^{-\beta\hbar\omega})\,d\omega = \frac{3N}{\omega_{max}^3}\int_0^{\omega_{max}}\omega^2\frac{\partial}{\partial\beta}\left[\ln(1-e^{-\beta\hbar\omega})\right]d\omega$$

$$= \frac{3N}{\omega_{max}^3}\int_0^{\omega_{max}}\omega^2\frac{1}{1-e^{-\beta\hbar\omega}}\cdot\hbar\omega e^{-\beta\hbar\omega}\,d\omega = \frac{3N}{\omega_{max}^3}\int_0^{\omega_{max}}\frac{\hbar\omega^3}{e^{\beta\hbar\omega}-1}\,d\omega = \langle E\rangle$$

b. Find C_V for low temperature limit. $k_BT << \hbar\omega_{max}$

$$\lim_{\hbar\omega_{max}\to\infty}\langle E\rangle = \frac{3N\hbar}{\omega_{max}^3}\frac{1}{(\beta\hbar)^4}\underbrace{\int_0^\infty\frac{(\beta\hbar\omega)^3}{e^{\beta\hbar\omega}-1}\,d(\beta\hbar\omega)}_{\frac{\pi^4}{15}} = \frac{3N(k_BT)^4}{\omega_{max}^3\hbar^3}\frac{\pi^4}{15} + \dots$$

$$C_V = \frac{\partial E}{\partial T} = \frac{4\pi^4 N k_B^4}{5\omega_{max}^3\hbar^3}\cdot T^3$$

for low temperatures.

$[C_V] = \frac{J}{K} = \frac{kg\,m^2}{s^2 K}$

$[k_B] = \frac{m^2 kg}{s^2 K}$ ✓

c. Now find C_v for high temperature limit $k_B T >> \hbar \omega_{max}$

$$\lim_{\hbar\omega_{max}\to 0} \langle E \rangle = \frac{3N}{\omega_{max}^3} \int_0^{\omega_{max}} \frac{\hbar\omega^3}{e^{\beta\hbar\omega} - 1} d\omega \qquad e^x \to 1 + x + O(x^2)$$

$$= \frac{3N}{\omega_{max}^3} \int_0^{\omega_{max}} \frac{\hbar\omega^3}{1 + \beta\hbar\omega - 1} d\omega = \frac{3N}{\omega_{max}^3} \int_0^{\omega_{max}} \frac{\hbar\omega^{3\,2}}{\beta\hbar\omega} d\omega$$

$$= \frac{3N}{\beta\omega_{max}^3} \underbrace{\int_0^{\omega_{max}} \omega^2 d\omega}_{\frac{1}{3}\omega^3 \Big|_0^{\omega_{max}}} = \frac{3N}{\beta\omega_{max}^3} \cdot \frac{1}{3}\omega_{max}^3 = N k_B T + \dots$$

$$C_v = \frac{\partial E}{\partial T} = N k_B \quad \text{for high temperatures} \qquad [C_v] = [k_B] \; \checkmark$$