QM-2 September 2019

Charged particle an electric field E charge q mass m

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 + qEx$$

$$[x, p] = i\hbar$$

a). For $E = 0$, Hamiltonian $H_0 = \hbar\omega(\hat{a}^+ \hat{a} + \frac{1}{2}) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$.
write down explicit forms of $\hat{a}$ & $\hat{a}^+$

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right) \Rightarrow (-ip + m\omega x)(ip + m\omega x) + \underbrace{m\omega\hbar}_{\text{from non-commutation}}$$

$$= \frac{1}{2m}\left[(-ip + m\omega x)(ip + m\omega x) + m\omega\hbar\right] = \hbar\omega\left[\frac{1}{2m\hbar\omega}\underset{\downarrow \, a^+}{(-ip + m\omega x)}\underset{\downarrow \, a}{(ip + m\omega x)} + \frac{1}{2}\right]$$

$$\hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega m}}(-ip + m\omega x) \qquad \hat{a} = \frac{1}{\sqrt{2\hbar\omega m}}(ip + m\omega x)$$

b. Find ground state and 1st excited state → functions, → energies for H_0

$$H_0 \Psi = \hbar\omega(\hat{a}^+\hat{a} + \tfrac{1}{2})\Psi = E\Psi$$

$$E_0 = \hbar\omega(\tfrac{1}{2} + \underbrace{\hat{a}^+ a)|0\rangle}_{0} = \frac{\hbar\omega}{2}$$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$
$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$|0\rangle = H_0\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-m\omega x^2/2\hbar}$$

$$= \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-m\omega x^2/2\hbar} \rightarrow \text{ground state.}$$

First excited state.

$$|1\rangle = H_1\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-m\omega x^2/2\hbar}$$

$$= \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\, x\, e^{-m\omega x^2/2\hbar}$$

$$E_1 = \hbar\omega(\tfrac{1}{2} + \hat{a}^+\hat{a})|1\rangle$$

$$= \hbar\omega(\tfrac{1}{2} + 1) = \frac{3}{2}\hbar\omega$$

c. Find change in ground state & 1st excited state energy for $\mathcal{E} \neq 0$

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \underbrace{\left(x^2 + \frac{2q\mathcal{E}}{m\omega^2} x + \left(\frac{q\mathcal{E}}{m\omega^2}\right)^2 \right)} - \frac{1}{2} m\omega^2 \left(\frac{q\mathcal{E}}{m\omega^2}\right)^2$$

$$\left(x + \frac{q\mathcal{E}}{m\omega^2}\right)^2 = \tilde{x}^2$$

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \tilde{x}^2 - \frac{1}{2}\frac{q^2\mathcal{E}^2}{m\omega^2} \quad \Rightarrow \quad \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \tilde{x}^2 = \underbrace{E + \frac{1}{2}\frac{q^2\mathcal{E}^2}{m\omega^2}}_{\tilde{E}}$$

$$\tilde{E} = \hbar\omega\left(n + \frac{1}{2}\right)$$

$$\tilde{E}_0 = \frac{\hbar\omega}{2} = E_0 + \frac{1}{2}\frac{q^2\mathcal{E}^2}{m\omega^2} \rightarrow E_0 = \frac{1}{2}\left(\hbar\omega - \frac{q^2\mathcal{E}^2}{m\omega^2}\right)$$

→ changes ground state by $-\frac{q^2\mathcal{E}^2}{2m\omega^2}$

$$\tilde{E}_1 = \frac{3}{2}\hbar\omega = E_1 + \frac{1}{2}\frac{q^2\mathcal{E}^2}{m\omega^2} \rightarrow E_1 = \frac{1}{2}\left(3\hbar\omega - \frac{q^2\mathcal{E}^2}{m\omega^2}\right)$$

→ changes 1st excited state by $\frac{-q^2\mathcal{E}^2}{2m\omega^2}$

→ all states will be shifted by a constant value of $\frac{-q^2\mathcal{E}^2}{2m\omega^2}$