QM-1 September 2020

2 bosons in infinite square well → indistinguishable.

$$V(x) = \begin{cases} 0 & x \in [0,a] \\ \infty & \text{otherwise} \end{cases}$$

weak interaction $V(x_1,x_2) = -aV_0(x_1-x_2)$

a. Ignore interaction. Find ground state & first excited state → wavefunctions → energies

$$-\frac{\hbar^2}{2m}\left[\nabla\psi_1 + \nabla\psi_2\right] + V\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_n^2}\psi_n = E_n\psi_n \quad \text{for 1 particle}$$

$$\frac{d^2}{dx_n^2}\psi_n = -\frac{2m}{\hbar^2}E_n\psi_n$$

$$\psi_n = A\sin(kx) + B\cos(kx) \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

$$\psi(a) = A\sin(ka) = 0 \qquad ka = \pi n \quad \text{for } n = 0, 1, \dots$$

$$\psi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right) \qquad k = \frac{\pi n}{a}$$

$$E = \frac{\pi^2 n^2\hbar^2}{2ma}$$

$$\psi_0 = \psi_1(x)\psi_2(x)$$

$$\psi_0(x_1,x_2) = \frac{2}{a}\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_2\right)$$

$$E_T = E_1 + E_2 = \frac{(\pi\hbar)^2}{2ma^2}(n_1^2+n^2) \Rightarrow E_0 = \frac{\pi^2\hbar^2}{ma^2}$$

First excited state $n_1 = 2$ & $n_2 = 1$ or $n_2 = 2$ & $n_1 = 1$

$$\psi_1(x_1,x_2) = A\left[\sin\left(\frac{2\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_2\right) + \sin\left(\frac{2\pi}{a}x_2\right)\sin\left(\frac{\pi}{a}x_1\right)\right]$$

$$E_T = E_1 + E_2 = \frac{\pi^2\hbar^2}{2ma^2}(1+2^2) = \frac{5}{2}\frac{\pi^2\hbar^2}{ma^2}$$

Normalize $\psi_1 \Rightarrow A = \frac{2}{a}\frac{1}{\sqrt{2}}$

$$\psi_1(x_1,x_2) = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{2\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_2\right) + \sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{2\pi}{a}x_2\right)\right]$$

b. Use first order perturbation theory to calculate effect on ground state & first excited state.

→ Both are non degenerate.

$V' = -aV_0\, \delta(x_1 - x_2)$

$$E_0' = \langle \psi_0 | V' | \psi_0 \rangle = \int_0^a \int_0^a \frac{4a}{a^2} \sin^2\left(\frac{\pi}{a}x_1\right) \sin^2\left(\frac{\pi}{a}x_2\right) \left(-V_0\, \delta(x_1 - x_2)\right) dx_1\, dx_2$$

$$= \int_0^a -\frac{4V_0}{a} \sin^4\left(\frac{\pi}{a}x_1\right) dx_1 = -\frac{4V_0}{a}\,\frac{3a}{8} = -\frac{3}{2}V_0$$

→ Ground state energy shift

First excited state.

$$E_1' = \langle \psi_1 | V' | \psi_1 \rangle = \int_0^a \int_0^a -\frac{2aV_0}{a^2} \left[\sin\left(\frac{2\pi}{a}x_1\right)\sin\left(\frac{\pi x_2}{a}\right) + \sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{2\pi}{a}x_2\right)\right]^2 \delta(x_1 - x_2)\, dx_1\, dx_2$$

$$= -\frac{2V_0}{a}\int_0^a \left[\sin\left(\frac{2\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_1\right) + \sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{2\pi}{a}x_1\right)\right]^2 dx_1 = -\frac{2V_0}{a}\int_0^a \left[2\sin\left(\frac{2\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_1\right)\right]^2 dx_1$$

$$= -\frac{8V_0}{a}\int_0^a \underbrace{\sin^2\left(\frac{2\pi}{a}x_1\right)\sin^2\left(\frac{\pi}{a}x_1\right)}_{\left\{\frac{1}{2}\left[\cos\left(\frac{2\pi}{a}x_1 - \frac{\pi}{a}x_1\right) - \cos\left(\frac{2\pi}{a}x_1 + \frac{\pi}{a}x_1\right)\right]\right\}^2 = \frac{1}{4}\left[\cos\left(\frac{\pi}{a}x_1\right) - \cos\left(\frac{3\pi}{a}x_1\right)\right]^2} dx_1$$

$$= -\frac{2V_0}{a}\left\{\int_0^a \left[1 - \sin^2\left(\frac{\pi}{a}x_1\right)\right] dx_1 - 2\int_0^a \frac{1}{2}\left[\cos\left(\frac{-2\pi}{a}x_1\right) + \cos\left(\frac{4\pi}{a}x_1\right)\right] dx_1 + \int_0^a \left[1 - \sin^2\left(\frac{3\pi}{a}x_1\right)\right] dx_1\right\}$$

$$= -\frac{2V_0}{a}\left[a - \frac{a}{2} - 0 - 0 + a - \frac{a}{2}\right] = -2V_0\left[2 - \frac{1}{2} - \frac{1}{2}\right] = -2\left[2 - 1\right] = -2V_0$$

$= -2V_0$

→ 1^{st} excited state energy shift.