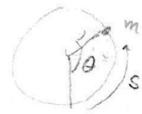
QM-2 September 2020

Particle m on ring with circumference L

a. Write down Hamiltonian & Schrödinger equation



$$H = \frac{\hat{P}^2}{2m}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{d\theta^2} = E\psi \quad \rightarrow \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{ds^2} = E\psi$$

$2\pi R = L$

$R = \frac{L}{2\pi}$

Boundary condition:

$\psi(\theta) = \psi(\theta + 2\pi)$ or $\psi(s) = \psi(s+L)$

b. Determine eigenstates & energy levels

$$\frac{d^2\psi}{ds^2} = -\frac{2m}{\hbar^2}E\psi$$

$\psi \sim A e^{iks}$ $\quad k = \sqrt{\frac{2mE}{\hbar^2}}$

$$\int_0^L A^2\psi^*\psi \, ds = \int_0^L A^2 ds = A^2 L = 1 \quad \Rightarrow \quad A = \frac{1}{\sqrt{L}}$$

$$\psi(s) = \frac{1}{\sqrt{L}} e^{i\sqrt{2mE}/\hbar \, s}$$

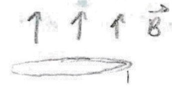
$$\psi(s) = \psi(s+L) \Rightarrow e^{iks} = e^{iks}e^{ikL} \Rightarrow e^{ikL} = 1 \Rightarrow kL = 2\pi n$$

$k = \frac{2\pi n}{L}$ for $n = 0, 1, 2, 3 \ldots$

$$\Rightarrow \frac{2\pi n}{L} = \sqrt{\frac{2mE}{\hbar^2}} \quad \Rightarrow \frac{2mE}{\hbar^2} = \frac{4\pi^2 n^2}{L}$$

$$E = \frac{2\pi^2\hbar^2 n^2}{mL}$$ for $n = 0, 1, 2 \ldots$

c. Assume particle has charge q. Find $\hat{H}$ & Schrodinger Equation.
Φ pierces ring.



Gauge transformation: $\vec{p} \to (\vec{p} - \frac{q}{c}\vec{A})$

$$\hat{H} = \frac{(\vec{P} - \frac{q}{c}\vec{A})^2}{2m}$$

Take $\vec{B} = B\hat{z}$

$$\hat{z}\cdot(\nabla \times \vec{A}) = \vec{B}\cdot\hat{z} \Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \frac{\partial}{\partial x}A_y - A_x\frac{\partial}{\partial y} = B$$

$\Rightarrow$ take $\frac{\partial}{\partial x}A_y = B \Rightarrow A_y = Bx$ $\qquad \vec{A} = [0, Bx, 0]$

$$\hat{H} = \frac{[P_x^2 + (p_y - \frac{q}{c}Bx)^2 + p_z^2]}{2m}$$

$\to$ Depends on x not y or z $\Rightarrow \Psi(x,y,z) = e^{ik_y y} e^{ik_z z} \chi(x)$

$\Rightarrow p_n = -i\hbar\frac{\partial}{\partial n}$

$$\hat{H}\psi = \frac{1}{2m}\left[p_x^2 + (\hbar k_y - \frac{q}{c}Bx)^2 + (\hbar k_z)^2\right]\psi = E\psi$$

$\to p_y\Psi = \hbar k_y\Psi$ & $p_z\Psi = \hbar k_z\Psi$

d. Find eigenstates & eigen energies.

$$\hat{H} = \frac{1}{2m}\left[p_x^2 + \alpha(x_0 - x)^2\right] = \left(E - \frac{\hbar^2 k_z^2}{2m}\right)$$

$x_0 = \frac{\hbar k_y c}{qB}$ $\qquad \alpha = (\frac{q}{c}B)^2$ $\qquad \tilde{E} = E - \frac{\hbar^2 k_z^2}{2m}$

$$\hat{H}\chi(x) = \frac{1}{2m}\left[p_x^2 + \alpha(x_0 - x)^2\right] = \tilde{E}\chi(x)$$

$$= \frac{1}{2m}p_x^2 + \frac{\alpha}{2m}\tilde{x}^2 = \tilde{E}\chi(x)$$

$\frac{\alpha}{2m} = \frac{1}{2}m\omega^2 \Rightarrow \frac{\alpha}{m} = m\omega^2$

$\omega^2 = \frac{\alpha}{m^2}$ $\qquad \omega = \frac{qB}{cm}$

$\chi(x) \to$ hermite polynomials

$$\tilde{E} = \hbar\omega(n + \tfrac{1}{2}) \Rightarrow E - \frac{\hbar^2 k_z^2}{2m} = \hbar\frac{qB}{cm}(n + \tfrac{1}{2})$$

$$E = \frac{\hbar^2 k_z^2}{2m} + \frac{\hbar qB}{cm}(n + \tfrac{1}{2})$$

B.C $\Rightarrow z = 0$

$x^2 + y^2 = r^2 = \frac{L^2}{4\pi}$

$$\Psi_{xyz} = C e^{ik_y y} e^{ik_z z} H_n e^{-(x_0 - x)^2 m\omega/2\hbar}$$

$$= C e^{ik_y y} H_n(x - x_0) e^{-(x_0 - x)^2 m\omega/2\hbar}$$