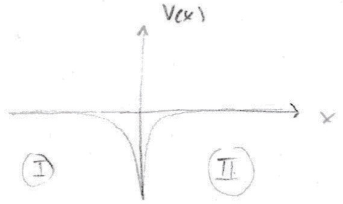
QM-3 September 2020

Particle mass m

Potential $V(x) = -V_0 \delta(x)$ $\quad (V_0 > 0)$



a Calculate E of bound states.
Write down wave functions

Region I:

$$\psi(x) = Ae^{kx}$$

Region II:

$$\psi(x) = Be^{-kx}$$

At $x=0$. $\quad \psi_I(x) = \psi_{II}(x) \Rightarrow A = B$

$$\psi_I = Ae^{kx} \qquad \psi_{II} = Ae^{-kx}$$

$$k = \frac{\sqrt{-2mE}}{\hbar}$$

Discontinuity in derivative:

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E - V(x))\psi(x)$$

$$\frac{\hbar^2 k^2}{2m} = -E$$

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{d^2\psi}{dx^2} = \int_{-\varepsilon}^{\varepsilon}\left[E - V(x)\right]\psi\, dx \qquad \varepsilon \to 0$$

$$\frac{d\psi_I}{dx} = Ake^{kx}$$

$$\frac{d\psi_{II}}{dx} = -Bke^{-kx}$$

$$-\frac{\hbar^2}{2m}\left\{\frac{d\psi}{dx}\Big|_{\varepsilon} - \frac{d\psi}{dx}\Big|_{-\varepsilon}\right\} = -V_0\psi(0)$$

$$\frac{d\psi_I}{dx}\Big|_{\varepsilon} - \frac{d\psi_{II}}{dx}\Big|_{-\varepsilon} = \frac{2mV_0}{\hbar^2}\psi(0)$$

$$Ak + Ak = \frac{2mV_0}{\hbar^2} \qquad \psi(0) = A$$

$$Ak = A\frac{mV_0}{\hbar^2}$$

$$\Rightarrow k = \frac{mV_0}{\hbar^2}$$

Normalization: $\int_{-\infty}^{\infty}\psi^*\psi\, dx = 1 \Rightarrow \int_{-\infty}^{0}|A|^2 e^{2kx}dx + \int_0^{\infty}|A|^2 e^{-2kx}dx \Rightarrow |A|^2\left[\int_{-\infty}^{0}e^{2kx}dx + \int_0^{\infty}e^{-2kx}dx\right]$

$$|A^2|\left\{\frac{1}{2k}e^{2kx}\Big|_{-\infty}^{0} + \frac{-1}{2k}e^{-2kx}\Big|_0^{\infty}\right\} = |A|^2\left[\frac{1}{2k} + \frac{1}{2k}\right] = |A|^2\frac{1}{k} = 1 \qquad |A|^2 = k \qquad A = \sqrt{k}$$

$$\psi_I(x) = \sqrt{\kappa}\, e^{\kappa x} \qquad \text{for } x < 0$$

$$\psi_{II}(x) = \sqrt{\kappa}\, e^{-\kappa x} \qquad \text{for } x > 0$$

$$\kappa = \frac{mV_0}{\hbar^2}$$

$$\psi(x) = \frac{\sqrt{mV_0}}{\hbar} \begin{cases} e^{mV_0 x/\hbar^2} & x < 0 \\ e^{-mV_0 x/\hbar^2} & x > 0 \end{cases}$$

Energy: $\kappa = \frac{\sqrt{2mE}}{\hbar} \Rightarrow \frac{\hbar^2 \kappa^2}{2m} = -E$

$$E = \frac{\hbar^2 m^2 V_0^2}{\hbar^4 2m} = \frac{-mV_0^2}{2\hbar^2}$$

b. Calculate reflection and transmission coefficients.

$$\psi_I = A e^{ikx} \rightarrow \psi_R = R e^{-ikx}$$

$$\psi_T = T e^{ikx}$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

For $x < 0$: $\psi = A e^{ikx} + R e^{-ikx}$ & for $x > 0$: $\psi = T e^{ikx}$

Continuous at $x = 0$: $A + R = T$

$$\left.\frac{d\psi}{dx}\right|_{\varepsilon} - \left.\frac{d\psi}{dx}\right|_{-\varepsilon} = \frac{2mV_0}{\hbar^2}\psi(0) \Rightarrow Tik - Aik + Rik = \frac{2mV_0}{\hbar^2}(A+R)$$

$$\beta = \frac{mV_0}{\hbar^2 k}$$

$$ik(T + R - A) = \frac{2mV_0}{\hbar^2}(A+R)$$

$$\rightarrow i(A + R + R - A) = \frac{2mV_0}{\hbar^2 k}(A+R) \rightarrow 2iR = \frac{2mV_0}{k\hbar^2}(A+R)$$

$$R(i - \beta) = A\beta \Rightarrow R = A\frac{\beta}{i-\beta}$$ → reflection amplitude.

$$T = A + A\frac{\beta}{i-\beta} = A\left[\frac{i - \beta + \beta}{i-\beta}\right] = A\left[\frac{i}{i-\beta}\right]$$ → transmission amplitude.

$$r = \frac{|R|^2}{|A|^2} = \frac{\beta^2}{(i-\beta)(-i-\beta)} = \frac{\beta^2}{1+\beta^2} = \left(\frac{mV_0}{\hbar^2 k}\right)^2 \frac{1}{1 + \left(\frac{mV_0}{\hbar^2 k}\right)^2}$$ → reflection coefficient

$$t = \frac{|T|^2}{|A|^2} = \left[\frac{i}{i-\beta}\right]\left[\frac{-i}{-i-\beta}\right] = \frac{1}{1+\beta^2} = \frac{1}{1 + \left(\frac{mV_0}{\hbar^2 k}\right)^2}$$ → transmission coefficient

$$t + r = \frac{\beta^2 + 1}{1 + \beta^2} = 1 \quad ✓$$